Selling Training Data*

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Abstract

In this paper, we develop a framework to analyze the design and price of supplemental training dataset. A monopolistic seller versions training datasets and associated tariffs to screen data buyers with different private datasets. The predictive power of private datasets in different states determine horizontal preference of the buyer while the overall predictive power determines vertical preference. The designer differentiates both the horizontal and vertical preference to trade-off the reduction of information rent and allocation efficiency. In the cases of binary situation, we provide an explicit construction of the four different schemes in the optimal menu. In the general case, two-tiered pricing mechanism is the optimal, where the buyers are sold a partially informative data in the first tier and a fully informative data in the second tier.
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1 Introduction

With increasing amounts of available dataset and rapidly advanced data processing techniques, data has played a central role in many economically significant situations. Data provides predictions and help decision makers to refine and optimize their actions, thus consequently improving her expected welfare. As digital technology advances and data collaboration becomes more frequent, businesses often integrate their existing data with supplemental dataset purchased from information intermediaries to improve prediction. Or they deliver the market data they collect of special use cases to an information broker, who in turn provides specialized decision support in combination with their more powerful data resources.

In this paper, we develop a framework to analyze the sale of supplemental dataset. In general, the pricing of these datasets often varies based on factors such as the size and specificity of the dataset, the level of data granularity, the intended use of the data, and the volume of data being purchased. Since the data buyer does not form a posterior belief based on its own data before purchasing, the seller's screening mechanism can only discriminate on different original datasets, which becomes the noteworthy difference from information selling.

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We formalize this problem in the classic Bayesian decision-theoretic model pioneered by Blackwell and the framework of mechanism design and information design. A single data buyer faces a decision problem under uncertainty, and a information seller maximizes his profits by designing the optimal menu of data with prices per buyer type. The value of data is measured by its incremental value in the decision problem.

The data buyer owns a private dataset containing information about a state variable that is relevant to his decision. Although the data buyer does not use the private dataset in advance, since the superimposed use of the data (transformed into an experiment) is equivalent to sequential use, the designer faces data buyers who have multiple beliefs, but cannot sell to their different beliefs separately. The revelation principle of the data buyer's private data tells that the posterior it produces is associated with the recommended actions. The key to selling training data is to first consider the beneficence of these recommended actions, which reflects the predictive power under different terminal decisions. When each state corresponds to a unique optimal action, for the private dataset, as if it is possible to distinguish the ability to predict each state.

The predictive power of the private dataset may be partial and different across states, which is private to the data buyer and unknown to the data seller. The predictive power of buyer's private dataset of some state determine his willingness to pay for any supplemental information of such state, resulting the differences in horizontal preference. And the overall predictive power also determine the vertical preference for more informative dataset.

Thus, from the perspective of the data seller, there are many possible types of data buyer with private multi-dimensional preference. Here we employ the recommendation mechanism, where the designer recommends an action profile to complement different possible prediction of the dataset in every state. The designer should finely design the information products to (i) utilize the differences in horizontal and vertical preference across types to maximize her revenue, and (ii) correlate the decision-relevant information for different states with the private predictive power in different states of the same type buyer.

Before giving an overview of our results, we emphasize three key attributes of data complicating the design of data sale mechanism. In the design of data selling mechanism, the seller can utilize two screening toolkits to achieve revenue-maximization: she allocates the data and decides its associated tariff to differentiate the menu, which resembles the quality and price control in classic one-dimensional mechanism design problem. However, it is not standard as it seems, three key attributes of data complicates this design problem, making it in nature a joint design of multi-dimensional allocation and information

1. Data provides different predictive power across states and is inherently multi-dimensional goods, thus making both the preference and the allocation goods multi-dimensional.

2. The recommendation mechanism employed in the data selling problem introduces the obedience/responsiveness constraints.

3. The complementarity and substitution between data differ, which means that even for the

same data, the utility function form may vary from buyers with different private data, thus distorting the analysis of mutual IC conditions.

We first investigate the structural properties of the optimal menu to reduce the dimensions and simplify the design problem. We first test the implementability of the predictions and restrict to those implementable predictions when combining with the private dataset. Then we apply the splitting lemma and demonstrate the structural welfare properties in data design. With structural welfare properties, we further show that the optimality imposes restrictions on the distortions in the data. Both the existence of the fully informative experiment and the constrained distortion to the experiments ("non-dispersed") in the optimal menu are guaranteed.

In the situation with binary type and binary state, we explicitly construct the four selling schemes in the optimal menu. The designer can utilize the multi-dimensional preference to implement product differentiation and price discrimination to extract surplus as much as possible. A direct device is the interaction between the the horizontal prediction power and the vertical prediction power of the two types. If the two-dimensional prediction power of the low type[1] is much stronger than or similar to the high type, then the "no-haggling" result for monopoly pricing in Riley and Zeckhauser (1983) applies. In other cases, the designer can always extract the valuation of the low type by selling dataset with partial informative prediction in another dimension when some dimension is no-haggling, and extract the information rent of the high type by exploiting the horizontal differences and versioning the dataset. And the designer trade-off the extraction of the efficient value of low type and the information rent of high type.

In the general case with continuous type in binary state, our main finding is that the optimal mechanism is simple but more flexible than the standard outcome of one-dimensional screening, even with the same predictive power in some dimension. In particular, it suffices to offer the buyers two-tiered pricing in the optimal selling policy: the types are partitioned into two tiers according to their predictive power of their private dataset; in the first tier, all types are sold a *dataset with partially informative prediction*, where the dimension with random predictive power is sold a null while the dimension with the same predictive power is sold a partially informative experiment, and charged a relatively low price; in the second tier, all types are sold a *dataset with fully informative prediction*, and charged a relatively high price. The threshold of the two tiers is determined similar to *the monopolist pricing*. By versioning the data for sale, the designer applies the no-haggling in one dimension and differentiation in another dimension.

From a methodological perspective, experiment as a private type in our framework gives rise to multi-dimensional private preference and the design of multi-dimensional allocation (of the accuracy of predictions). Moreover, the *selection* of payoff function form of an arbitrary experiment for some type is endogenous and thus type-dependent, which makes the anlaysis of IC conditions *type pair-dependent*. With the experiment as private type, we apply the splitting lemma to derive the implications in welfare of common prior in this setting. To analyze the type pair-dependent IC, we utilize a novel approach by defining two functions representing the tightness of IC and

[1] Here the low type is the one whose private dataset provides more accurate prediction, thus lowering his willingness to pay.

responsiveness and exploit their structural properties. And then we transform all these conditions into tractable constraints in a standard optimization problem. Finally, we apply the novel solution technique based on a general extension of Carathéodory's theorem found in Kang (2023), which has many applications in recent papers in mechanism design and information design [2] to solve it. These techniques may be useful in other settings with type pair-dependent mutual IC or joint design of information and mechanism.

Related Literature. Our work is related to the emerging literature on designing information and data selling mechanism to imperfectly informed decision makers. Previous literature mainly focus on the design and price of append information (Admati and Pfleiderer (1986); Admati and Pfleiderer (1990); Babaioff et al. (2012); Bergemann and Bonatti (2015); Bergemann et al. (2018))[3]. There are also some papers focusing on the setting where the agent can endogenize the information acquisition beyond purchasing additional information. Li (2022) discovers that endogenous information distorts the incentives, and reduces the revenue of the information seller. Our paper focuses on the price and design of training data. A key difference lies in that the varied and systematic way of combining the predictions of data broadens the scope for versioning and differentiating, compared to the information selling. The combination of predictions from datasets is under the minimization of statistical error, and thus endogenous and varying from the buyers.

Our research also contributes to the multi-dimensional mechanism design[4]. . Compared to standard multi-dimensional screening and bundling (Adams and Yellen (1976); McAfee et al. (1989); Armstrong and Rochet (1999); Carroll (2017); Haghpanah and Hartline (2021); Yang (2022); Yang (2023)), the design of data selling mechanism is the joint design of multi-dimensional goods allocation under multi-dimensional preference and information. Thus both the interactions between the dimensions and the introduction of responsiveness constraints jointly complicates the analysis. We explore the trade-off between the constraints and identify the optimal structure of the constraints using some new-defined functionals, and then transform it into a standard one-dimensional screening problem, providing a novel approach to tackling the joint design of information and multi-dimensional allocation problem.

The most relevant works to ours are Bergemann et al. (2018) and Bergemann et al. (2022). The designers all pose a contract at the ex ante stage to screen the buyer with private information/data. In Bergemann et al. (2018) and Bergemann et al. (2022), they consider the selling of *input data*, like cookie and purchase history, to buyers with private input data. In their setting, data buyers have private information before contracting. For example, the decision makers (like lenders, advertisers, and health care providers in Bergemann et al. (2018)'s examples) may have prior interactions with the subjects (like the borrower, specific consumers, patient's family), which is relevant to their decisions. While in our setting, buyers with private dataset purchase *training data* to complement

[2]see, for example, Fuchs and Skrzypacz (2015); Bergemann et al. (2018); Loertscher and Muir (2023); Dworczak and Muir (2024) in mechanism design and Le Treust and Tomala (2019); Doval and Skreta (2022)in information design, see Dworczak and Muir (2024) for more details

[3]See Bergemann and Bonatti (2019) for a more comprehensive survey.

[4]Our analysis also intersects with the mechanism design concerning the strategic disclosure of pricing information, as explored by Lizzeri (1999); Ottaviani and Prat (2001); Bergemann and Pesendorfer (2007); Eső and Szentes (2007); Krähmer and Strausz (2015). These studies delve into the ex ante design of a disclosure rule and a pricing policy.

her private one, in order to train algorithms to predict outcomes and recognize patterns. For example, the purchased medical data is used for training supervised learning models to predict disease diagnosis or drug efficacy. The purchased financial data is used for the stock price prediction or credit scoring. The buyers does not have specific knowledge relevant to their decisions before contracting and appending the training data.

From a technical perspective, in Bergemann et al. (2018), the buyer receivers some private interim belief purchases the information structure. In other words, the designer allocates multi-dimensional goods to screen buyers with one-dimensional but incongruent preference order. In binary state, the design problem is indeed one-dimensional. While in our setting, the private type of the buyer is Blackwell experiment. And the preference, resulting from the predictive power of the data in different dimensions, is in nature multi-dimensional. In Bergemann et al. (2018), they propose some structural properties of the optimal menu to reduce the dimensions. We derive similar conclusions by proving an almost different approach , because in our setting, the designer should recommend an action profile to all possible interim beliefs while their designer only need to recommend one action to some posterior. Moreover, their model can be reduced to a standard one-dimensional allocation under one-dimensional preference with some additional constraint . While in our setting, even fixing one dimensional preference, it can not be reduced to one-dimensional allocation due to the interaction between ex-ante responsiveness and incentive compatibility.

And finally, although both apply the general extension of Carathéodory's theorem[5], we derive different structures of the optimal menu. In Bergemann et al. (2018), the responsiveness constraint, joint with the standard monotonicity constraint, constitute the constraint of the allocation goods. Therefore, they give an upper bound of the cardinality of the optimal menu. However, in our setting, the responsiveness constraint and the monotonicity constraint respectively constrains the allocation in two dimension. Therefore, we derive a sharper prediction of the structure of the optimal menu.

Outline. The rest of the paper is organized as follows. Section 2 presents the setup of the data selling mechanism design and further explores the structural property in the data selling mechanism design. Section 3 report our results in the binary situation. In Section 4, we generalize the binary situation and explore the structure of the optimal mechanism. In Section 5, we state the direction of our future work. The proof of Lemmas and Theorems can be found in the Appendix.

[5]In their work, they apply the Fundamental Theorem of Linear Programming by discretion and then extend the conclusion to a continuous form by considering the a converging (in distribution) discrete distribution, which is in fact equivalent to ours.

2 Basic Model

2.1 Model and Notation

Decision Under Uncertainty. A single data buyer with private statistical experiment faces a decision problem under uncertainty. The state of world ω is drawn from a finite set $\Omega = \{\omega_1, \ldots, \omega_i, \ldots, \omega_I\}$. The data buyer chooses an action a from a finite set $A = \{a_1, \ldots, a_j, \ldots, a_J\}$. The ex post utility is denoted by $u(\omega_i, a_j) \triangleq u_{ij} \in \mathbb{R}_+$. Following , we assume $I \leq J$ and $u_{ii} > u_{ij}$, $\forall j \neq i$. These two assumptions capture the idea that the action space is at least as rich as the state space and that for every state ω_i, there is a unique action (labeled a_i) that maximizes the decision maker's utility in that state.

Denote $\mathcal{U} = \{u_{ij}\}_{I \times J}$ the $I \times J$ payoff matrix in the decision problem. A useful special case is one in which the data buyer faces binary ex post payoffs in each state, i.e., he seeks to match the state and the action. In that case, the utility function $u(\omega_i, a_j)$ is then given by $u(\omega_i, a_j) \triangleq \mathbb{I}_{[i=j]} \cdot u_i$, and $u_i \triangleq u_{ii}$. This formulation assumes that, in each state, the data buyer assigns the same value to each wrong action. This value is normalized to 0 because adding a state-dependent translation to the utility function does not affect preferences over actions. Under this assumption, it is without loss of generality to assume that the sets of actions and states have the same cardinality: $|A| = |\Omega| = I = J$.

Table 1: Payoff Matrix

u	a_1	$\cdots$	a_J
ω_1	u_{11}	$\cdots$	u_{1J}
$\vdots$	$\vdots$	$\ddots$	$\vdots$
ω_I	u_{I1}	$\cdots$	u_{IJ}

Table 2: Matching Utility

u	a_1		a_I
ω_1	1		0
$\vdots$		$\ddots$	
ω_I	0		1

Experiment. The private statistical experiment $E'_n = (S, \pi'_n)$ (equivalently, an information structure) is the type of the data buyer[6][7], where players share the same signal space $S' = \{s'_1, ..., s'_K\}$, and the likelihood functions of signal $\pi'_{nik} \equiv Pr[s_k|\omega_i]$ are type dependent. Note that $\sum_{k=1}^{K} \pi'_{ik} = 1$ for all i. From the perspective of the data seller, these private experiments are distributed according to a distribution $F(n) \in \Delta(N)$, which we take as a primitive of our model.

In line with the interpretation of selling supplemental information, the beliefs can be generated from a common prior and privately observed signals after the state is realized. Thus, suppose there is a common prior $\mu \in \Delta(\Omega)$. The decision maker privately observes a signal and every signal induces a posterior. Indeed, we can assume that agent‘s type is the distribution of posteriors, i.e. $\{\mu_{n1}, ..., \mu_{nK}\}$ with $\Pr(\mu_{nk}) = \sum_{i=1}^{I} \mu_i \pi'_{nik}$ for all $k \in 1, ..., K$, where $\mu_{nk} \in \Delta(\Omega)$ and $\mu_{nki} \equiv \mu_{nk}(\omega_i) = \frac{\mu_i \pi'_{nik}}{\sum_{i'=1}^{I} \mu_{i'} \pi'_{ni'k}}$.

[6]An interesting point is that, in the sense of conclusions, this setup is nearly equivalent to the setup where $\{\kappa_i\}_{i \in I}$ is the private type of the buyer, where κ_i is the matching utility in state ω_i, i.e. $\kappa_i = u_{ii}$ and $\kappa_i \in [0, 1]$.

[7]We can also define the measure of predictive power of the private dataset as the private type (as in the section 3. Here we use more standard setup to directly apply the splitting lemma to derive some other properties in the following parts.

The data buyer seeks to augment his initial private information by obtaining additional information from the data seller in order to improve the quality of his decision making. The seller designs statistical experiments $E = (S, \pi)$ with prices to maximize his profits, where $S = \{s_1, ..., s_R\}$ and $\pi_{ir} \equiv Pr[s_r|\omega_i]$.

We assume throughout that the realization of the buyer's private signal s'_k and that of the signal s_r from any experiment E sold by the seller are independent, conditional on the state ω. In other words, the buyer and the seller draw their information from independent sources.

Value of Experiement. We first describe the value of the buyer's initial information and then determine the incremental value of an experiment $E = (S, \pi)$. Type n data buyer's decision problem is given by conditioning on accepting signal s'_k, choosing an optimal action $a\left(s'_k \mid E_n\right)$ that maximizes the expected utility (of states) given the posterior, i.e.,

$$a\left(s'_k \mid E_n\right) \in \arg\max_{a_j \in A}\{\textstyle\sum_{i=1}^{I} \mu_{nki} u_{ij}\} \text{ and } u(s'_k \mid E_n) \triangleq \max_j\{\textstyle\sum_{i=1}^{I} \mu_{nki} u_{ij}\}.$$

The expected utility of type n is therefore given by

$$U_n \triangleq \textstyle\sum_{k=1}^{K} \Pr(\mu_{nk}) u(s'_k \mid E_n) = \sum_{k=1}^{K} \max_j \left\{\sum_{i=1}^{I} \mu_i \pi'_{nik} u_{ij}\right\}.$$

By contrast, if the data buyer has access to an experiment $E = (S, \pi)$, he observes the private signal s'_k and the bought signal realization $s_r \in S$, updates his beliefs, and then chooses an appropriate action. Consequently, for any signal $s_r \in S$ that occurs with strictly positive probability π_{ir}, an action that maximizes the expected utility of type n is given by

$$a(s_r, s'_k|E_n) \in \arg\max_{a_j \in A}\left\{\textstyle\sum_{i=1}^{I}\left(\frac{\mu_{nki}\pi_{ir}}{\sum_{i'=1}^{I}\mu_{nki'}\pi_{i'r}}\right) u_{ij}\right\}$$

which leads to the following conditional expected utility:

$$u(s_r, s'_k|E_n) \triangleq \max_j \left\{\textstyle\sum_{i=1}^{I}\left(\frac{\mu_{nki}\pi_{ir}}{\sum_{i'=1}^{I}\mu_{nki'}\pi_{i'r}}\right) u_{ij}\right\}.$$

Integrating over all signal realizations s_k and subtracting the value of prior information, the (net) value of an experiment E for type n is given by $V(E, n)$, where

$$V(\mathrm{E}, n) \triangleq u(\mathrm{E}, n) - U_n = \sum_{k=1}^{K}(\sum_{r=1}^{R} \max_j \left\{\sum_{i=1}^{I} \mu_i \pi'_{nik} \pi'_{ir} u_{ij}\right\} - \max_j \left\{\sum_{i=1}^{I} \mu_i \pi'_{nik} u_{ij}\right\}). \quad (1)$$

Mechanism. By the revelation principle, the menu of experiments can be described as $\mathcal{M} = \{E_n, t_n\}_{n \in N}$, where E_n is the experiment for type n and $t_n \in \mathbb{R}_+$ is its associated tariff. Our goal is to characterize the revenue-maximizing menu for the seller. The timing of the game is as follows:

1. the seller posts a mechanism $\mathcal{M}$
2. the type n buyer chooses an experiment E_n and pays the corresponding price t_n
3. the true state ω is realized
4. the buyer receive two signals, one from his private experiment E'_n, another from the experiment E_n she bought, and she chooses an action a
5. payoffs are realized

2.2 Joint Design of Mechanism and Information

The seller's choice of a revenue-maximizing menu of experiments may involve, in principle, designing one experiment per buyer type. We denote the indirect net utility for the truth-telling agent by $V_n \triangleq V(\mathrm{E}, n) - t(n)$.

The seller's problem consists of maximizing the expected transfers

$$\max_{\{E(n),t(n)\}} \int_{n\in N} t(n)dF(n)$$

subject to incentive-compatibility constraints (IC)

$$V_n \geq V\left(E(n'), n\right) - t(n'), \quad \forall n, n' \in N,$$

and individual-rationality constraints (IR)

$$V_n \geq 0, \quad \forall n \in N.$$

To simplify the seller‘s problem, we can further reduce the set of menus of experiments to a smaller and very tractable class, thus canceling out the max operator in $V(E, n)$, the value of arbitrary experiment for some type n (as in 1). We can without loss of generality restrict our attention to the mechanisms such that if the buyer reports truthfully, following the recommendation from the seller maximizes her expected payoff. In the data selling problem where the private type of buyer is statistical experiment, each experiment entails a type-dependent mapping from signals into action recommendation profiles.

$E(n)$ is responsive if every signal $s \in S$ leads type n to a different optimal action vector. Precisely speaking, by constructing an onto mapping from the signal to the recommendation outcome, we can w.l.o.g represent the signal as a recommendation profile for all possible posteriors, i.e $s_r = (a_{r1}, ..., a_{rK})$, where $a_{rk} \in A$. An experiment E is responsive for some agent if and only if the optimal action for this agent conditional on accepting s_r and private signal s'_k is a_{rk} for all $s_r \in S$ and $s'_k \in S'$, and, in particular

$$a(s_r, s'_k|E_n) = a_{rk}, \quad \forall s'_k \in S', s_r \in S.$$

And a responsive menu is a menu in which experiment E_n is responsive for agent n for all n. A direct observation is that the maximal cardinality of a responsive menu is J^K.

Importantly, responsiveness is only required for every experiment $E(n)$ and for the corresponding type n. In other words, we do not require this condition to hold if signals in $E(n)$ are evaluated by a different type $n' \neq n$. Finally, we define an outcome of a menu as the joint distribution of states, actions, and monetary transfers resulting from every type's optimal choice of experiment and subsequent choice of action.

Lemma 1 establishes that it is without loss of generality to restrict attention to responsive menus—direct revelation mechanisms in which every experiment $E(n)$ is responsive[8].

Lemma 1 (Revelation Principle in Mechanism with Blackwell Experiment)**.** *The outcome of every menu $\mathcal{M} = \{\mathcal{E}, t\}$ can be attained by a responsive menu.*

We can merge the signals inducing the same recommendation profile to derive a responsive menu and the merging is implementable and strictly better for the designer. The responsive agent's valuation for the merged experiment is unchanged, while other agents' valuation decrease by Blackwell Theorem because the merging is a garbling. So both IC and IR constraints are relaxed and the designer can get weakly better.

The implementability of recommendation lies in the likelihood ratios between different states of the signals.To illustrate this point, suppose that there exist two posteriors for one agent. The posterior 1 believes that relative to ω_1, ω_2 is more likely to be realized, and converse for posterior 2. It is difficult to implement a signal to induce posterior 1 choosing a_2 while posterior 2 choosing a_1. Therefore we can further reduce the cardinality of the signal space.

Notice that in binary state situation, we can reorder the index of posteriors as below:

$$\frac{\mu_{n11}}{\mu_{n12}} \geq \frac{\mu_{n21}}{\mu_{n22}} \geq ... \geq \frac{\mu_{nK1}}{\mu_{nK2}} \text{ for all } n$$

So we can only implement recommendation profile s_r in which $s_{rj} = a_1$ implies that $s_{ri} = a_1$ for all $i \leq j$. Denote the set of these recommendation profiles as $\mathcal{A}^*$

Lemma 2 (Implementable Signals in Binary State)**.** *In binary state, the support of the recommendation profiles in a responsive experiment is a subset of $\mathcal{A}^*$*

Relabel the recommendation profile such that $a^1 = (a_1, a_1, ..., a_1)$, $a^2 = (a_1, a_1, ..., a_2)$, ..., $a^{K+1} = (a_2, a_2, ..., a_2)$. So $\mathcal{A}^* = \{a^1, ..., a^{K+1}\}$. A direct observation is that the cardinality of recommendation profile reduces to $|\mathcal{A}^*| = K + 1$.

Formally, under matching utility and binary state, the responsive/obedience constraints of signal k for agent n is[9]

[8] The proof is motivated form Bergemann et al. (2018) and we modify it to apply in our setting.

[9] We can also assume general payoff matrix to get similar results. The assumption of matching utility is only to omit some unnecessary algebra.

$$\frac{\mu_{n(K+1-k)2}}{\mu_{n(K+1-k)1}} \leq \frac{\pi_{1k}}{\pi_{2k}} \leq \frac{\mu_{n(K+2-k)2}}{\mu_{n(K+2-k)1}} \text{ for all } k = 1, 2, ..., K+1$$

where $\frac{\mu_{n02}}{\mu_{n01}} = 0$ and $\frac{\mu_{n(K+1)2}}{\mu_{n(K+1)1}} = +\infty$

2.3 Structural Property of the Optimal Menu

After the test of implementability and responsiveness, we have successfully reduce the dimensions of the parameters for design. Now we further explore the structural property of the experiments in the optimal menu to reduce more dimensions.

It is immediate that we can restrict our attention to the menu where the fully informative experiment $\overline{E}$ lies in. If not, choose the one charged the highest fee and replace its menu with the fully informative experiment and a weakly higher fee[10].

Lemma 3 (Existence of the Fully Informative Experiment)**.** *Every outcome of the optimal menu can be attained by one menu where $\overline{E}$ lies in the optimal menu.*

Define recommendation profile a^{k_1} Pareto dominant the a^{k_2} in state ω_i as $a^{k_1} \gtrsim_{\omega_i} a^{k_2}$, which means that in state i, a^{k_1} recommends at least one more posteriors to choose the optimal action a_i who are recommended to choose a_j in a^{k_2}, without changing recommendations to other posteriors. Then, $a^1 \gtrsim_{\omega_1} a^2 \gtrsim_{\omega_1} ... \gtrsim_{\omega_1} a^{K+1}$ and $a^{K+1} \gtrsim_{\omega_2} a^K \gtrsim_{\omega_2} .. \gtrsim_{\omega_2} a^1$.

For $i = 1, ..., I$ and $j_1, j_2 \in \{1, ..., K+1\}, j_1 \neq j_2$, denote $a^{j_1} \xrightarrow{i} a^{j_2}$ and $E(a^{j_1} \xrightarrow{i} a^{j_2}) = (\pi', S)$ as the adjustment of $E = (\pi, S)$ as followed: $\pi'_{ij_1} = \pi_{ij_1} - \delta$, $\pi'_{ij_2} = \pi_{ij_2} + \delta$ and others unchanged. Considering the adjustment in responsive experiment E for agent n, denote the valuation change under $a^{j_1} \xrightarrow{i} a^{j_2} (i = 1, ..., I)$, $\Delta(a^{j_1} \xrightarrow{i} a^{j_2} \mid E, n) \triangleq \frac{V(E(a^{j_1} \xrightarrow{i} a^{j_2}), n) - V(E, n)}{\delta}$.[11]

Lemma 4 (Properties of Structural Welfare Adjustment)**.** *For any responsive experiment E for type n, we have the following structural welfare adjustment properties:*

1. $\Delta(a^{j_1} \xrightarrow{i} a^{j_2} \mid E, n) = -\Delta(a^{j_2} \xrightarrow{i} a^{j_1} \mid E, n)$
2. *the following adjustment always brings constant welfare change*
   (a) $\Delta(a^{K+1} \xrightarrow{1} a^1 \mid E, n) = \mu_1$ *and* $\Delta(a^1 \xrightarrow{2} a^{K+1} \mid E, n) = \mu_2$
   (b) *there exists $k^\star$ such that* $\Delta(a^{K+1} \xrightarrow{1} a^{k^*+1} \mid E, n) + \Delta(a^1 \xrightarrow{2} a^{k^*} \mid E, n) = u_n$

Statement 1 is obvious. Adjustments in statement 2 is to "restore" the valuation with fully informative experiment/without information. The second outcome of the adjustment in statement 2 stems from the common prior and the splitting lemma. In fact, both the two adjustments can be viewed as the systematic minimization of the statistical error, resulting in constant decrease in the disutility.

[10]The proof is from Bergemann et al. (2018).

[11]In fact, the adjustment is dependent on δ, for simplicity, we omit it because it is useless for our analysis (or assume it is finitely small)

An experiment is non-dispersed if and only if $\pi_{1K+1} = 0$ and $\pi_{21} = 0$. A menu is shared-responsive if and only if all experiments in the menu are responsive for all types.

Table 3: Non-Dispersed Experiment

	a^1	$\cdots$	a^{K+1}
ω_1	π_{11}	$\cdots$	0
ω_2	0	$\cdots$	π_{2K+1}

Table 4: Experiment to Design

E	(a_1, a_1)	(a_1, a_2)	(a_2, a_2)
ω_1	$1 - \pi_2$	π_2	0
ω_2	0	π_1	$1 - \pi_1$

The structural welfare adjustment can further guarantee the non-dispersed property of the experiments in the optimal responsive menu.

Lemma 5 (Structures of the Optimal Experiment)**.** *In the optimal experiment.*

1. *Every experiment is non-dispersed*
2. *With binary state, every outcome of optimal menu can be attained by a shared responsive one.*

With binary state,the seller designs the likelihood function of recommendation profiles $a^1 = (a_1, a_1)$, $a^2 = (a_1, a_2)$, $a^3 = (a_2, a_2)$. With the structural properties above, we can transform the seller‘s problem into an explicit constrained optimization problem.

3 Binary Situation

Considering the situation with binary state, binary action, binary type, common prior $\mu = (\frac{1}{2}, \frac{1}{2})$[12] and uniform type distribution $\Pr(\text{type } i) = \frac{1}{2}$, $i = 1, 2$. To simplify the notation, denote the two type as:

$$
\begin{aligned}
\mu &= \left(\frac{1}{2}, \frac{1}{2}\right) \\
&= \frac{\mu_2 - \frac{1}{2}}{\mu_1 + \mu_2 - 1} (\mu_1, 1 - \mu_1) + \frac{\mu_1 - \frac{1}{2}}{\mu_1 + \mu_2 - 1} (1 - \mu_2, \mu_2) \\
&= \frac{\mu_2' - \frac{1}{2}}{\mu_1' + \mu_2' - 1} \left(\mu_1', 1 - \mu_1'\right) + \frac{\mu_1' - \frac{1}{2}}{\mu_1' + \mu_2' - 1} \left(1 - \mu_2', \mu_2'\right)
\end{aligned}
$$

where $(\mu_1, 1 - \mu_1)$ and $(1 - \mu_2, \mu_2)$ are the posteriors of the type 1, and $(\mu_1', 1 - \mu_1')$ and $(1 - \mu_2', \mu_2')$ are of the type 2. Suppose $\mu_1, \mu_1', \mu_2, \mu_2' > \frac{1}{2}$ without loss of generality by the splitting lemma.

Denote (θ_1, θ_2) as the portion of the total information value obtained from the posterior $(\mu_1, 1 - \mu_1)$ and posterior $(1 - \mu_2, \mu_2)$, respectively. More specifically, the multiplier of the market share of the corresponding posterior and the valuation for the fully informative one of that posterior, i.e.

[12] The main conclusions can be naturally extended to $\mu = (p, 1 - p)$ with more algebra.

$$\theta_1 = \underbrace{\frac{\mu_2 - \frac{1}{2}}{\mu_1 + \mu_2 - 1}}_{\text{market share}} \underbrace{(1-\mu_1)}_{V(\overline{E},(\mu_1,1-\mu_1))} \quad , \quad \theta_2 = \underbrace{\frac{\left(\mu_1 - \frac{1}{2}\right)}{\mu_1 + \mu_2 - 1}}_{\text{market share}} \underbrace{(1-\mu_2)}_{V(\overline{E},(1-\mu_2,\mu_2))}$$

and similarly define (θ_1', θ_2') associated with the type 2 buyer.

We call θ_1 the predictive power on state ω_1 of the private data owned by a data buyer. The rationale is that θ_1 measures the incremental decision value for the fully information under the posterior where the data owner is optimal in believing the state is ω_1 and choosing a_1 without the information. This predictive power is measured in terms of the information value for decision-making, rather than a statistical prediction accuracy. Compared with (μ_1, μ_2), (θ_1, θ_2) more directly reflects the buyer's valuation and preference for information. We use (θ_1, θ_2) and (θ_1', θ_2') to replace (μ_1, μ_2) and (μ_1', μ_2') as the private type of the buyer since they are well-corresponded,

We also suppose that the type 1 is the H-igh type and type 2 is the L-ow type in the sense of the valuation for the fully informative experiment, i.e. $V(\overline{E}, 1) \geqslant V(\overline{E}, 2)$, or $\theta_1 + \theta_2 \geqslant \theta_1' + \theta_2'$ where $\overline{E}$ is the fully informative experiment

Table 5: Fully Informative Experiment

$\overline{E}$	(a_1, a_1)	(a_1, a_2)	(a_2, a_2)
ω_1	1	0	0
ω_2	0	0	1

Table 6: Experiment to Design

E	(a_1, a_1)	(a_1, a_2)	(a_2, a_2)
ω_1	$1-\pi_2$	π_2	0
ω_2	0	π_1	$1-\pi_1$

The private training data set can be understood as the predictability of the two state. The higher predictability for state ω_i, the lower the willingness to pay for state i. Thus we can explicitly figure out the interactions between multi-dimensional preference in this screening problem using θ_i and θ_i'. The vertical valuation, i.e. valuation for the fully informative experiment as a whole, can be represented as $V(\overline{E}, 1) = \theta_1 + \theta_2$, while the horizontal valuation θ_1 and θ_2 represents the valuation for the fully informative experiment of the two posteriors.

By the existence of the fully informative experiment and the results above, the designer only needs to design the responsive one for low type, which is also responsive for the high type.

The valuation for this experiment is

$$\begin{aligned} V\,(\mathrm{E}, 1) &= \frac{1-\pi_1}{2} + \frac{1-\pi_2}{2} + \frac{(\mu_2 - \frac{1}{2})\mu_1\pi_1}{\mu_1+\mu_2-1} + \frac{(\mu_1 - \frac{1}{2})\mu_2\pi_2}{\mu_1+\mu_2-1} - \frac{\left(\mu_2 - \frac{1}{2}\right)\mu_1}{\mu_1+\mu_2-1} - \frac{\left(\mu_1 - \frac{1}{2}\right)\mu_2}{\mu_1+\mu_2-1} \\ &= \theta_1 + \theta_2 - \theta_1\pi_1 - \theta_2\pi_2, \\ V\,(E, 2) &= \theta_1' + \theta_2' - \theta_1'\pi_1 - \theta_2'\pi_2. \end{aligned}$$

Notice that the value of the experiment is linear with respect to the predictive power. From the perspective of marginal experiment for different posterior realizations, π_{12} misleads the $(1-\mu_2, \mu_2)$ and π_{22} misleads the $(\mu_1, 1-\mu_1)$ (as in Table 10). They are damage goods only for one dimension of the horizontal preference, motivating us to relabel the coordinate of the likelihood function as in Table 6. Therefore, from the expression of experiment value, the predictive power happens to become a quality preference for harmful misleading signals in the designed information matrix.

The responsiveness constraint is:

$$k_1 \equiv \max\left\{\frac{\theta_2}{\frac{1}{2}-\theta_1}\frac{\theta_2'}{\frac{1}{2}-\theta_1'}\right\} = \max\left\{\frac{1-\mu_2}{\mu_2}, \frac{1-\mu_2'}{\mu_2'}\right\} \leqslant \frac{\pi_1}{\pi_2} \leqslant \min\left\{\frac{\mu_1}{1-\mu_1}, \frac{\mu_1'}{1-\mu_1'}\right\} = \min\left\{\frac{\frac{1}{2}-\theta_2}{\theta_1}, \frac{\frac{1}{2}-\theta_2'}{\theta_1'}\right\} \equiv k_2.$$

Moreover, the responsiveness constraint of E for type 1 buyer (similar for type 2 buyer) can also be represented as a linear production possibility set of the damage goods in the following reduced form:

$$\theta_1\pi_1 + \theta_2\pi_2 \leq \min\{\tfrac{1}{2}\pi_1, \tfrac{1}{2}\pi_2\}.$$

The economic interpretations behind the three components in responsiveness is about statistical error. $\theta_1\pi_1 + \theta_2\pi_2$, $\frac{1}{2}\pi_1$, and $\frac{1}{2}\pi_2$ are respectively the loss of error when choosing (a_1, a_2), (a_1, a_1) and (a_2, a_2) when receiving the recommendation (a_1, a_2). Responsiveness requires that combining the prediction of the private dataset E_{t_1} and the purchased dataset E_1, the prediction of the purchased dataset always bring the minimized statistical error and would be obeyed.

Now the designer‘s problem is:

$$\max_{E,t_H,t_L} \frac{1}{2}\left(t_H + t_L\right)$$

s.t.

$$\begin{aligned}
&V(\overline{E},1) - t_H \geqslant 0 && \text{(IR-H)}\\
&V(E,2) - t_L \geqslant 0 && \text{(IR-L)}\\
&V(\overline{E},1) - t_H \geqslant V(E,1) - t_L && \text{(IC-H)}\\
&V(E,2) - t_L \geqslant V(\overline{E},2) - t_H && \text{(IC-L)}\\
&k_1 \leqslant \tfrac{\pi_1}{\pi_2} \leqslant k_2 && \text{(Responsiveness)}
\end{aligned}$$

It is not hard to see that $t_H \geq V(\overline{E}, 2) \geq t_L$ considering the optimality of the mechanism. Thus, the IC-L is always not binding. Then we can immediately derive that IR-L is binding. Let $T = t_H + V(E,2) - V(\overline{E},1)$, the designer‘s problem can be reduced as

$$\max_{E,T} \quad T$$

s.t.

$$\begin{aligned}
&T \leqslant \theta_1'\left(1-\pi_1\right) + \theta_2'\left(1-\pi_2\right) && \text{(IR-H)}\\
&T \leqslant \left(2\theta_1' - \theta_1\right)\left(1-\pi_1\right) + \left(2\theta_2' - \theta_2\right)\left(1-\pi_2\right) && \text{(IC-H)}\\
&\max\left\{\frac{\theta_2}{\frac{1}{2}-\theta_1}\frac{\theta_2'}{\frac{1}{2}-\theta_1'}\right\} = k_1 \leqslant \frac{\pi_1}{\pi_2} \leqslant k_2 = \min\left\{\frac{\frac{1}{2}-\theta_2}{\theta_1}, \frac{\frac{1}{2}-\theta_2'}{\theta_1'}\right\} && \text{(Responsiveness)}
\end{aligned}$$

Define

$$\Delta = \text{RHS(IR-H)} - \text{RHS(IC-H)} = (\theta_1 - \theta_1')(1-\pi_1) + (\theta_2 - \theta_2')(1-\pi_2) \tag{2}$$

where Δ measures the information rent for the H-type in this screening problem. Notice that when the predictable power in two states of low type are both stronger than the ones of high type, i.e. $(\theta_1, \theta_2) > (\theta_1', \theta_2')$, the information rent can never be eliminated. Otherwise when there exists one dimension where the predictive power of the low type is weaker than the high type, i.e. $\theta_1 < \theta_1'$ or $\theta_2 < \theta_2'$, the information rent can always be eliminated with proper allocation of (π_1, π_2) [13]. In this case, the optimal menu results in both IC-H and IR-H binding, which provokes novel trade-off in mechanism design.

It is also clear that the basic trade-off in the data selling problem is the extraction of information rent versus the extraction of efficient low type valuation, in the the reduced form of the designer‘ problem. the designer allocates the damage goods π_1 and π_2 to screen the low type, which also results in the disutility of statistical error and reduces the extraction of the efficient valuation of low type.

Here we state the main result in the binary setting. As figure 1 shows, the optimal selling mechanism involves the interaction of vertical differences, internal horizontal differences and external horizontal differences, resulting in four typically different information selling schemes.

Theorem 1 (Optimal Selling Mechanism)**.** *When low type (θ_1', θ_2') lies*

1. *in the zone I, the seller implements no discrimination policy, i.e. selling $\overline{E}$ to both types.* $((\pi_1, \pi_2) = (0,0)$, $t_H = t_L = \theta_1' + \theta_2')$
2. *in the zone II*[14]*, the seller implements exclusive policy, i.e. only selling $\overline{E}$ to the high type.* $((\pi_1, \pi_2) = (1,1)$, $t_H = \theta_1 + \theta_2$, $t_L = 0)$
3. *in the zone III-(i), the seller implements partial discrimination policy, i.e. selling $\overline{E}$ to type-H, and a fixed experiment E to type-L.* ($\Delta > 0$,$\pi_{3-i} = 1$, $\frac{\pi_1}{\pi_2} = k_i$, IR-(L) is binding)
4. *in the zone IV-(i), the seller implements full discrimination policy, i.e. selling $\overline{E}$ to type-H, and E to type-L.* ($\Delta = 0$, $\frac{\pi_1}{\pi_2} = k_i$, IR-(H) and IR-(L) are binding)

Proof. See the Appendix. □

3.1 Economic Interpretations of Theorem 1

Zone I and II. In zone I and zone II, the optimal selling policy resembles the no-haggling result of Riley and Zeckhauser (1983). In zone I, the data seller implements no discrimination policy, selling

[13] See details in the Appendix

[14] The coefficient of the boundary line is determined by the market share between the high type and the low type. Here the same market share induces the $\frac{1}{2}$

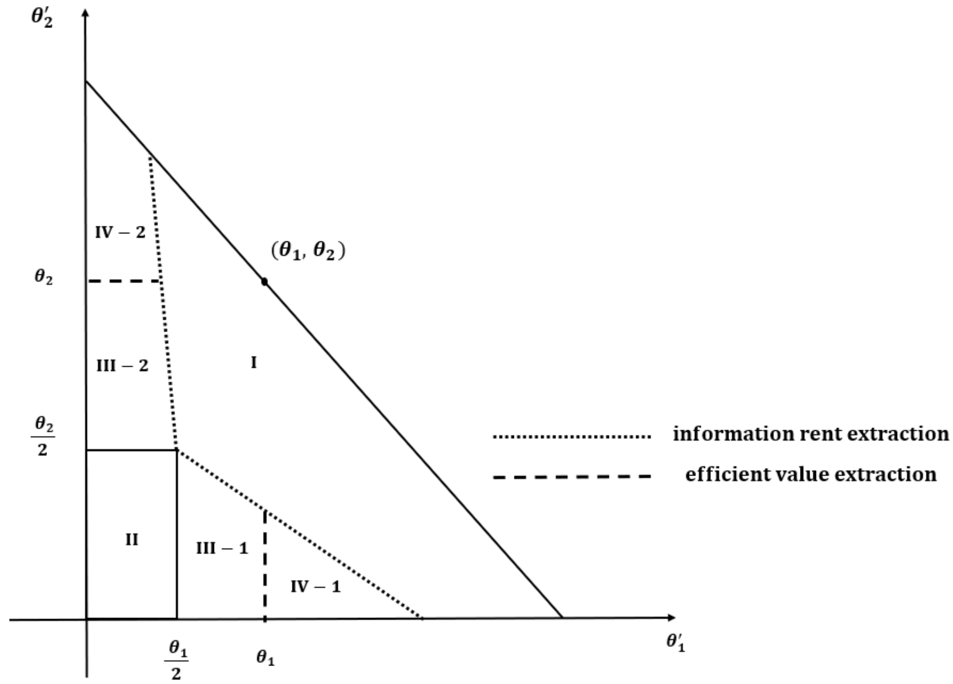

Figure 1: Optimal Selling Schemes

fully informative experiment Ē to both types. While in zone II, the seller implements exclusive policy and only sells Ē to high type[15]. Notice that in the two zones, both the horizontal and vertical valuations of the low type are either similar to or much smaller than the high type ($\theta_1'+\theta_2' \ll \theta_1+\theta_2$ or $\theta_1' + \theta_2' \approx \theta_1 + \theta_2$), so the two-dimensional preference can be approximately reduced to an one-dimensional preference in these situations. We can therefore appeal to the no-haggling result of Riley and Zeckhauser (1983) that establishes the optimality of an extremal policy. Such a policy consists of either allocating the object (here, the information) with probability 1 ($\pi_1 = 0$, $\pi_2 = 0$, $t_H = t_L = \theta_1' + \theta_2'$) or not allocating it at all ($\pi_1 = 1$, $\pi_2 = 1$, $t_H = \theta_1 + \theta_2$, $t_L = 0$) .

Zone III. In zone III, the seller implements partial discrimination, selling $\overline{E}$ to type-H, and a constant E to type-L, such that (IR-L),(IC-H),(Responsiveness) are binding. In zone III, one valuation of the low type is much smaller than the high type while another is slightly smaller. The designer can partially differentiate the low type one to (i) fully extract the valuation of the low type and (ii) (incompletely) reduce the information rent of high type, i.e. $\Delta > 0$.

Explicitly speaking, in the zone III-1, the low type gets the experiment in the form of table 7, where π_1^* is a constant decided by the obedience constraint parameter k_1. The sold experiment is equivalent to sell the two "marginal" experiments to the two posterior realizations in the form of table 8.

In the dataset for the low type, the dimension with relatively strong predictive power of type-

[15]It is noteworthy that the zone II always exists and its measure is dependent on the market share between the high type and the low type, which is different from traditional non-haggling result in binary situation.

Table 7: Experiment for (θ_1', θ_2')

E	(a_1, a_1)	(a_1, a_2)	(a_2, a_2)
ω_1	0	1	0
ω_2	0	π_1^*	$1 - \pi_1^*$

Table 8: Marginal Experiment for θ_1', θ_2'

θ_1'	a_1	a_2
ω_1	1	0
ω_2	π_1^*	$1 - \pi_1^*$

θ_2'	a_1	a_2
ω_1	0	1
ω_2	0	1

L is sold null, and the weak one gets a partial informative experiment fully revealing one state. Assuming that a seller can design information on two independent dimensions of predictive power, then based on the predictions of zone I and II, the seller will sell information only to the high type on the dimension where the buyer is more different, and sell complete information to both sides on the dimension where the buyer is less different. However, such sales require the design of E satisfying $\pi_1 = 0, \pi_2 = 1$, which violates the responsiveness conditions and makes recommendation (a_1, a_2) infeasible. Constrained by the likelihood ratio when recommending different actions to two posteriors, the seller sacrifices some extraction in the dimension with less difference, providing as little information as possible to maintain buyer's acceptance of the recommendation.

This interpretation raises the next question: why does the optimal menu is selected by reducing the information provided to the less diverse dimensions to meet the responsiveness condition, but not by increasing the amount of information provided to the more diverse dimension, or a combination of the two ways? We have proved that when the low type buyer has weaker predictive ability in both dimensions, no matter how the experiment is designed, the high type buyer has a positive information rent, i.e. $\Delta > 0$. This means that IC-H must be binding, and IR-H must be not, so that only IC-H and responsiveness constraints limit the tariff that can be obtained from high-type buyers. Responsiveness constraints determine the optimal ratio of π_1 to π_2, while IC-H associates with the shadow price of π_1 to π_2 satisfying that ratio. Depending on whether the shadow price negative or positive under given type parameters, the seller chooses to maximize or minimize, respectively, (π_1, π_2) while maintaining the ratio, which results in two corresponding results: $(\pi_1^*, 1)$ and $(0, 0)$.

Moreover, the boundary between zone I and III-1 characterizes the above trade-off. When the dimensions with large difference in predictive power are relatively close, the high type itself does not have much information rent, and seller efficiently extract the surplus of the low type by reducing (π_1, π_2). On the contrary, when the dimension with large difference in predictive power is not close enough, the information rent generated by the high type in this dimension is higher. By increasing (π_1, π_2), the information is only sold to the low type in the dimension with small difference, avoiding screening in the more diverse dimension, resulting in a significant decrease in the information rent paid.

A more interesting point is that in the whole III-1, only one fixed experiment E is sold. Naturally, E does not change with θ_2, because sellers do not sell information to low types in this dimension, but counterintuitively, the design of E does not change with respect to different θ_1. This result comes from the shared responsive property in Lemma 5. Because the high-type has lower overall predictive power, it is more difficult to recommend (a_1, a_2), which means his responsiveness constraint is tighter. Therefore, it is essentially responsive constraint of the high type that determines the selection of π_1, causing it not to vary with the parameters of the lower type. If the designer only considers the responsive conditions of the low type to determine a lower π_1, it will lead to the high type making decision (a_1, a_1) in facing the signal (a_1, a_2) when disguised as the low type, which provides information for the posterior corresponding to θ_2, thus increasing the benefit of dishonesty. This results in paying higher information rents for high types and thus less profits.

Zone IV. In zone IV, the seller implements full discrimination, selling $\bar{\text{E}}$ to type-H, and E to type-L, which smoothly changes in this zone, such that (IR-L),(IC-H),(IR-H),(Responsiveness) is binding. When one valuation of the low type is much smaller than the high type while another is slightly higher. The designer can fully differentiate the low type one to (i) extract the valuation of the low type in two dimensions and (ii) fully extract the information rent of high type.

Precisely speaking, the experiment sold to the low type is in the form of table 9, where π_1^* and π_2^* change smoothly in (θ_1', θ_2'). The sold experiment is equivalent to sell the two "marginal" experiments to the two posterior realizations in the form of table 10.

Table 9: Experiment for (θ_1', θ_2')

	(a_1, a_1)	(a_1, a_2)	(a_2, a_2)
ω_1	$1 - \pi_2^*$	π_2^*	0
ω_2	0	π_1^*	$1 - \pi_1^*$

Table 10: Marginal Experiment for θ_1', θ_2'

θ_1'	a_1	a_2
ω_1	1	0
ω_2	π_1^*	$1 - \pi_1^*$

θ_2'	a_1	a_2
ω_1	$1 - \pi_2^*$	π_2^*
ω_2	0	1

Notice that two dimensions of the low type are both sold a partial informative experiment fully revealing one state. Meanwhile, the amount of information provided by the designer increases as the predictive power of the low type decreases in either dimensions. As before, the boundary between zone IV and zone I also reflects the trade-off between extracting surplus from the low type and reducing the information rent paid to the high type.

The most important intuition of zone IV is that it reflects the interaction of design screening in the two dimensions of predictive power. As figure 2 shows, even with fixed θ_2', the posterior corresponding to θ_2' would get sold more informatively as θ_1' increases. On the one hand, when horizontally transitioning from zone III to zone IV, the optimal menu changes from selling no information to this dimension to selling partial information. On the other hand, within zone IV, the amount of information sold on this dimension also increases as the level of predictive power on the other dimension decreases.

The reason for this interaction is that, the experiment designed by the seller will now result in a positive or negative information rent for the high-type buyer, which never happen in one-dimensional mechanism design problems. When the designer in zone IV continues to sell the low-type buyer only the information on the dimension with less difference in predictive power, it is noted that as now $\theta_1 < \theta_1'$, in this scheme which regresses to this one-dimension, the low-type buyer becomes the "high type" instead. This results in tighter IR constraint than IC constraint for the high-type, and thus the tariff that can be charged for high types are reduced. By moving

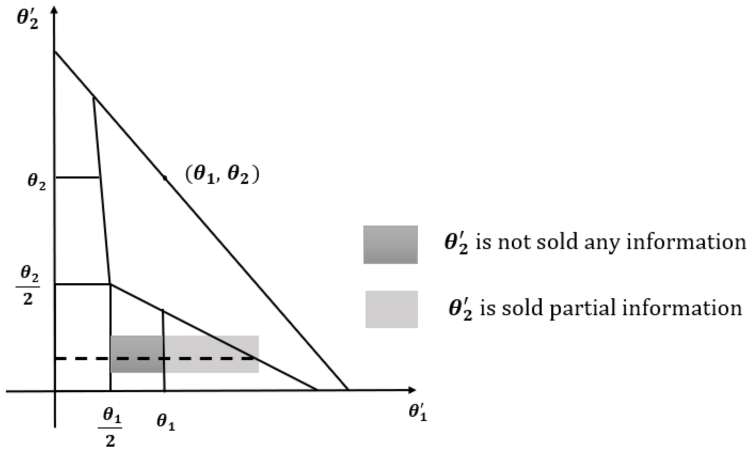

Figure 2: Zone III versus Zone IV

the experiment designed for the lower types along the recommendation boundary to provide more information in both dimensions, the unique experiment that makes IC-H and IR-H both binding (i.e. the information rent is zero) could be found. This design maximized the surplus that could be extracted from the high-type buyer.

In the design and price of information, Bergemann et al. (2018) also finds that the designer can properly allocate the information products to weigh the (IC-H) and (IR-H) conditions and make both of them binding. But the economic drivers behind ours are different from theirs. In Bergemann et al. (2018), the private preference of the buyers is in nature one-dimensional but may not be congruent in orders. However, in our setting, the private preference of the buyers is in nature multi-dimensional.

To summarize, in the data selling mechanism, the designer can utilize the multi-dimensional preference to implement product differentiation and price discrimination to extract surplus as much as possible. A direct device is the interaction between the horizontal value ((θ_1, θ_2) and (θ_1', θ_2')) and vertical value ($\theta_1 + \theta_2$ and $\theta_1' + \theta_2'$) of the two types. Moreover,due to multi-dimensional nature of the problem, both the external and the internal differences in horizontal preference play a central role in the design.

In fact, the responsiveness of the experiment implies that the designer designs a data with three kinds of predictions: (i) state 1 prediction (ii) pooling prediction (iii) state 2 prediction. Only the pooling prediction brings disutility from the statistical error when combining the predictions of the private dataset. Therefore. the key for differentiated design and price discrimination lies in the pooling prediction, which complements the private experiment in such way that both posteriors adopt their default choices. And the external and the internal differences in horizontal preference shape the the space for designing this prediction.

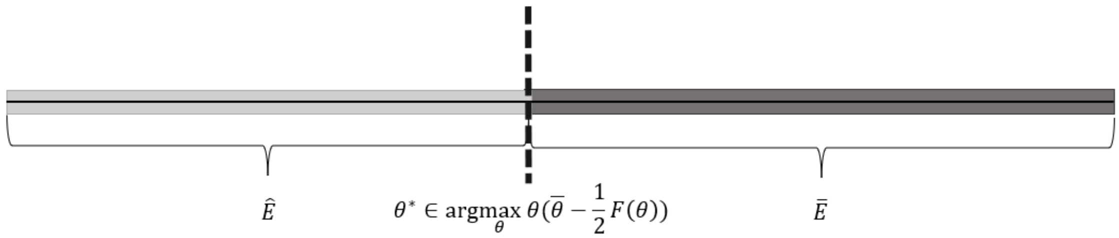

Figure 3: Optimal Selling Schemes (General Case)

4 General Case

We now complete the analysis of the binary-action environment. The analysis of the binary situation demonstrates that the selling data is in nature a multi-dimensional screening problem. To tackle the multi-dimensional screening problem, we utilize our structural properties of welfare adjustments to reduce the dimensions.

In the general case, we consider the private preference for data as (θ_1, θ_2), where $\theta_1 = m$ is a constant and $\theta_2 = \theta$ is a random variable with the support $\Theta = [\underline{\theta}, \overline{\theta}] = [0, \frac{1}{2} - m]$[16] and the distribution is $F(\theta)$. Therefore the agent‘s private type is θ.

Notice that in this situation, the non-dispersed property still holds. Therefore the designer only needs to design the $\{\pi_1(\theta), \pi_2(\theta)\}$ for all $\theta \in \Theta$, satisfying the incentive compatibility, individual rationality and responsiveness.

We begin by stating our main result in the general case, which characterizes the structure of the optimal selling menu.

Table 11: Experiment for the First Tier

$\hat{E}$	(a_1, a_1)	(a_1, a_2)	(a_2, a_2)
ω_1	0	1	0
ω_2	0	$\frac{\theta^*}{\overline{\theta}}$	$1 - \frac{\theta^*}{\overline{\theta}}$

Table 12: Experiment for the Second Tier

$\overline{E}$	(a_1, a_1)	(a_1, a_2)	(a_2, a_2)
ω_1	1	0	0
ω_2	0	0	1

Theorem 2 (The Optimality of the Cutoff Mechanism)**.** *The optimal selling mechanism is*

1. $(E_\theta, t_\theta) = (\overline{E}, t_{\theta^*})$ *for all for* $\theta \in [\theta^*, \overline{\theta}]$
2. $(E_\theta, t_\theta) = (\hat{E}, \hat{t})$ *for* $\theta \in [\underline{\theta}, \theta^*)$*, where* $\pi_1 = \frac{\theta^*}{\overline{\theta}}$*,* $\pi_2 = 1$
3. $\theta^* \in \arg\max_\theta \theta \left(\overline{\theta} - \frac{1}{2}F(\theta)\right)$

As theorem 2 shows, the optimal mechanism takes a simple and economically interpretable structure. As in figure 3, it suffices to offer the buyers two-tiered pricing in the optimal selling

[16]It will derive a symmetric conclusion in the converse case where θ_1 is a r.v and θ_2 is a constant.

policy: the types are partitioned into two tiers according to their predictive power of their private dataset; in the first tier, all types are sold a partially informative experiment $\hat{E}$, where the θ_2 dimension are sold a null while the θ_1 dimension is sold a partially informative experiment, and charged a relatively low price; in the second tier, all types are sold a fully informative experiment $\overline{E}$ and charged a relatively high price. The threshold of the two tiers is determined similar to the monopolist pricing[17].

4.1 Discussion of Theorem 2

An intuitive (and ideal) revenue-maximization selling policy is that $\operatorname{im} \pi_2 \subseteq \{0, 1\}$ and $\pi_1 = 0$, where the designer implements no-haggling in the dimension with random predictive power and extracts all the efficient valuation in another dimension by selling $\pi_1 = 0$ (i.e. sells fully informative prediction to this dimension).

Unfortunately, it can not be implemented because the two dimensions are interconnected and it is not equal to simple aggregation even when the prediction power in one dimension is fixed, due to the existence of pooling prediction. As figure 4 shows, when discussing the value of experiment to some other type (e.g. E_θ to θ), the pooling prediction (a_1, a_2) may implement different recommendations when combining the prediction of the private dataset, such as (a_1, a_1) or (a_2, a_2), for other types. When the prediction of (a_1, a_2) is not credible at all (i.e. $\pi_1 = 0$), even the buyer with the weakest prediction power in that dimension, will confound the prediction when combining the prediction of her private dataset. The consequence is that the valuation of the data is now completely dependent on π_1, i.e. the statistical error under the prediction (a_1, a_1) (as in the left table in the second line). The no-haggling policy of the designer does not work anymore and eventually the ideal policy can not implemented.

As the figure 4 shows, the statistical error caused by the data in the optimal menu, is from the false prediction in state ω_i, i.e. π_{-i}. If the predictive power of the private dataset is too weak to confound the prediction with other predictions, it will induce statistical error only dependent on one parameter π_i. Conditional on fixing one dimension, the designer always wants to extract the efficient value of that dimension, thus reducing the false prediction parameter of that dimension low enough to not induce any confounding prediction (a_2, a_2). Given this, the confounding prediction can only be (a_1, a_1), of which statistical error is type-dependent for the one with weak enough predictive power in the varying dimension. In this sense, data is equivalent to the goods with utility in monetary units, which is irrelevant to the prediction of the private dataset. And purchasing training data is equivalent to paying monetary transfer. The mutual incentive compatibility requires the designer to equalize the monetary transfer of the data, which pins down the allocation of π_1 and separates the design of π_1 and π_2.

Another natural problem is the optimality of the two-tiered pricing. And we discuss it from the perspective of economics and mathematics.

[17] The coefficient of the $F(\theta)$ is dependent on the prior. The whole conclusion can be naturally extended to the prior $(\mu, 1-\mu)$ with a little tedious algebra.

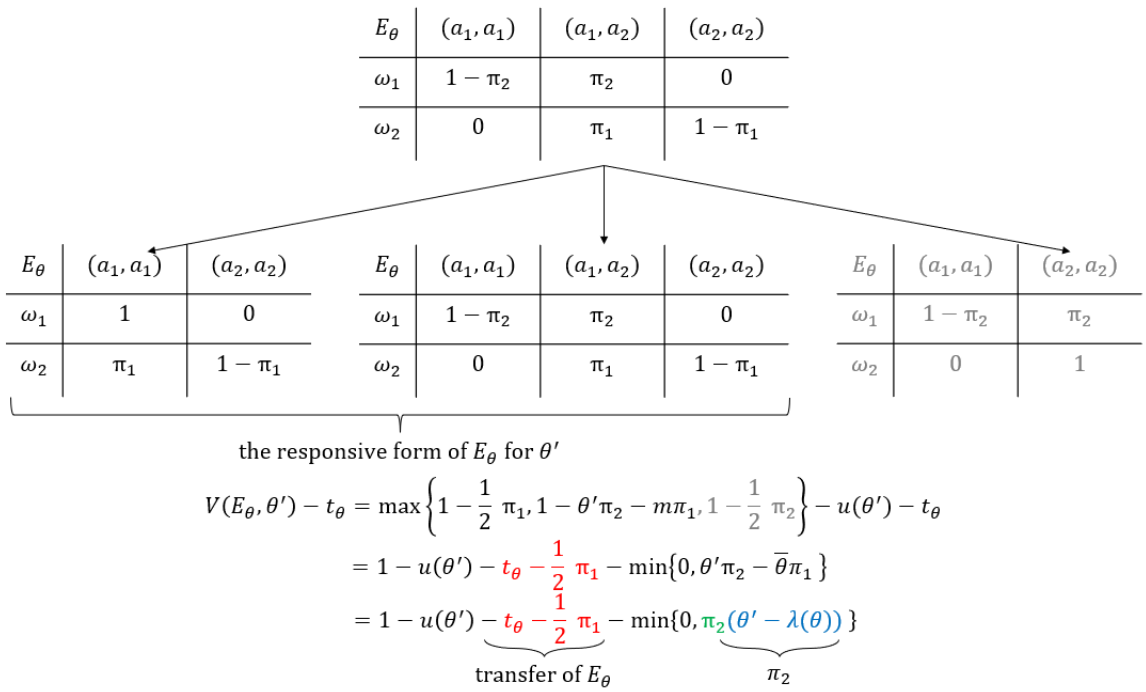

Figure 4: Responsive Form of E_θ

Economically, a two-tier pricing mechanism is attractive because it not only allows the designer to utilize the multi-dimensions to differentiate the buyers in a flexible way, thus extracting the information rent, but also guarantees the extraction of the efficient valuation of the low type (i.e. whose dataset has more accurate predictions). The designer can also reduce the information rent of the buyers with high valuation by differentiating the fixed dimension, which reduces the extraction of the efficient valuation in that dimension. The cardinality of the optimal tiers reflects the trade-off between the extraction of the efficient valuation of the low type and the information rent of the high type.

Mathematically, with the trade-off between the responsiveness and incentive compatibility, we can derive the structure of the optimal menu. It turns out that the either the responsiveness in the structure of the constraints is not binding, or the responsive one is sold the fully informative experiment in the optimal menu. Meanwhile, the linearity of the problem implies that either the responsiveness is binding, or the incentive compatibility is binding.

With the structure of the binding constraints, we can successfully separate the interaction between dimensions and between IC and responsiveness, and apply the approaches in the standard one-dimensional screening problem. The responsiveness shapes the allocation in the fixed dimension while the monotonicity shapes the allocation in varying dimension. The cardinality of the options in the menu is determined by the Carathéodory's theorem. The allocation of $\pi_2(\theta)$ is an extreme point of the set of non-increasing function from Θ to $[0,1]$. And the allocation of $\pi_1(\theta)$ is pinned down by the envelope theorem and responsiveness.

The last problem is the determination of the threshold θ^*. The no-haggling gives birth to the form similar the monopolist pricing. And the differentiation in the fixed dimension gives birth to the coefficient of the distribution in the formula, which is decided by the common prior in our setting and makes it different from the standard threshold in no-haggling. Compared to one-dimensional screening, the designer adjusts the threshold to trade-off the extraction in two dimensions.

To summarize, the multi-dimensions broadens the scope of differentiation and the designer can sell informative data to the relatively high dimension of the low type to extract both the information rent of the high type and the valuation of the low type. The designer trade-offs the extraction of the information rent of the high type and the efficient value of the low type. Also note that theorem 2 pins down the structure of the optimal menu without imposing any regularity assumptions on the distribution.

In the remainder of this section, we sketch the proof of Theorem 2. The proof overview casts some light on how the interaction between responsiveness and incentive compatibility complicates the data selling problem, and how to utilize the different responsive form when combining predictions to differentiate the buyers. It also helps justify the setup in the general case by showing how the structure of the constraints are pinned down by the primitives of the model.

4.2 Proof of Theorem 2

We proceed by first analyzing the trade-off between IC and Responsiveness constraints, then characterizing the structure of the optimal menu, and finally transform the designer's problem with an explicit and approachable functional optimization problem and uses an infinite-dimensional extension of Carathéodory's theorem to solve it.

Step 1: Analyzing the Trade-off between IC and Responsiveness

In the general case, the responsiveness condition for θ is:

$$\frac{\theta}{\overline{\theta}} \leqslant \frac{\pi_1(\theta)}{\pi_2(\theta)} \leqslant \frac{\frac{1}{2}-\theta}{\frac{1}{2}-\overline{\theta}} \quad \text{(Responsiveness)}$$

We call $[\frac{\theta}{\overline{\theta}}, \frac{\frac{1}{2}-\theta}{\frac{1}{2}-\overline{\theta}}]$ as the responsive zone. It is immediate to derive that the responsive condition becomes strictly more stringent if θ increases. A direct corollary is that if one experiment is responsive for some type θ, then it is responsive for all $\theta' < \theta$.

First the responsiveness can be further reduced to $\frac{\pi_1(\theta)}{\pi_2(\theta)} \in [\frac{\theta}{\overline{\theta}}, 1]$ for all θ with structural adjustment, as in figure 5. Conditional on $\theta_1 = Constant$, the preference for the information of state ω_1 is constant among all types. Therefore, the value of π_1 is either type-independent and can be regarded as the same monetary transfer for those or zero. If $\pi_1(\theta) > \pi_2(\theta)$, the disutility of statistical error for buyer is either the same or zero because they never choose (a_1, a_1) when combining the prediction of their private dataset. Thus the designer can always charge a strictly higher fee without violating any constraints by decreasing π_1.

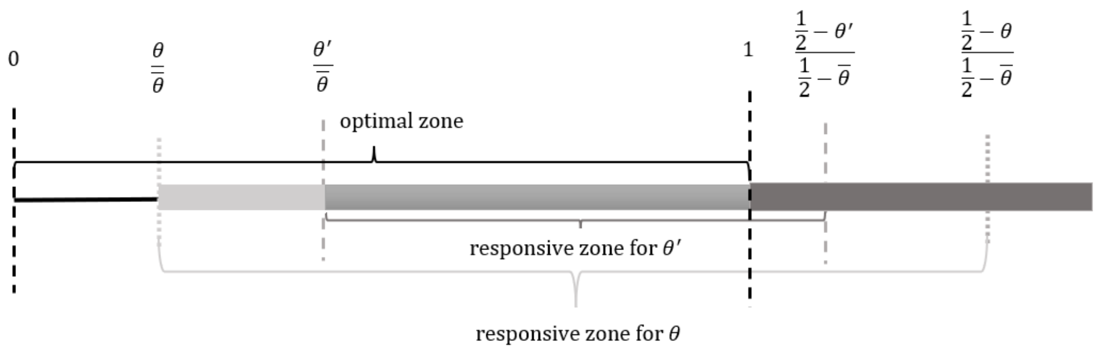

Figure 5: Responsive Zone

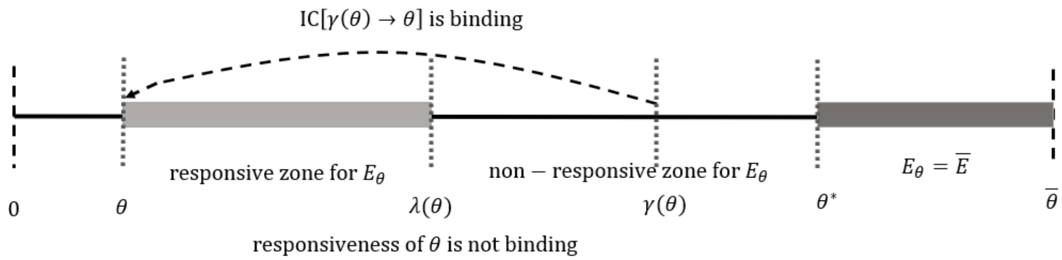

Figure 6: Trade-off between IC and Responsiveness

Define the function $\lambda(\theta) : \Theta \to \Theta$ as below: (i) $\lambda(\theta) = \overline{\theta}\frac{\pi_1(\theta)}{\pi_2(\theta)}$ if $\pi_2(\theta) \neq 0$ (ii) $\lambda(\theta) = \overline{\theta}$ otherwise. $\lambda(\theta) \in \left[\theta, \overline{\theta}\right]$. $\lambda(\theta)$ is to identify the threshold of the responsiveness of E_θ[18]. The experiment E_θ is responsive for $\theta' \in [\theta, \lambda(\theta)]$, and pools the recommendation profile (a_1, a_2) with (a_1, a_1) for $\theta' \in \left[\lambda(\theta), \overline{\theta}\right]$. Moreover, the responsiveness of θ is binding if and only if $\lambda(\theta) = \theta$.

We now analyze the trade-off between IC and responsiveness in the design problem. Denote IC[$\theta \to \theta'$] as the IC condition that type θ is unwilling to pretend as θ'.

Lemma 6. *In the optimal mechanism, there always exists $\theta^* \in \Theta$,*

1. *$E_\theta = \overline{E}$ if and only if $\theta \geq \theta^*$*
2. *for all $\theta < \theta^*$, $\theta < \lambda(\theta)$ and there exists $\theta' > \lambda(\theta)$,* IC[$\theta' \to \theta$] is binding

The first statement in lemma 6 implies that $\overline{E}$ is sold with a cut-off selling policy. It is from the motonicity in the informativeness of the private prediction in the responsive data goods for the informativeness of the prediction, and its equivalence constant monetary transfer when non-responsive. The fully informative experiment both feature the two properties. Joint with the mutual IC between two types with different levels of prediction power, we can derive this statement.

The second statement consists of two components. First, for all types under the threshold for selling data with fully informative predictions, their responsiveness are never binding. Second,

[18]Here we assume the responsiveness is not binding for any $\theta \neq \overline{\theta}$ allocated with $\overline{E}$.

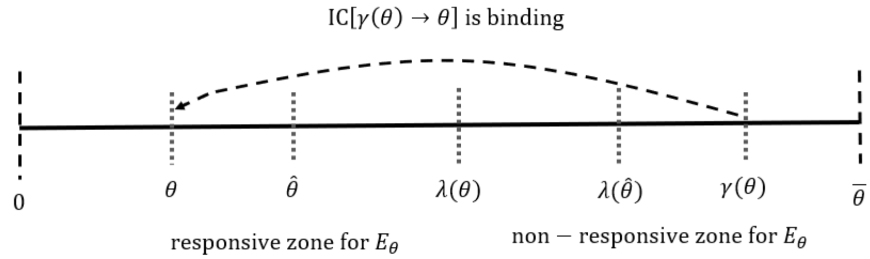

Figure 7: Properties of $\lambda(\theta)$ and $\gamma(\theta)$

there always exists some type non-responsive for the predictions in their data, the type is indifferent between her menu and the one's.

The second statement is from interaction between the tightness of responsiveness and incentive compatibility in the optimal menu. With the structural adjustment, at least one of them should be satisfied in the optimal menu. And if the responsiveness is not binding, we can always find a non-responsive one indifferent between her menu and the non-binding one's.

Step 2: Characterizing the Structure of the Optimal Mechanism

The standard analysis of mutual IC in data selling problem is choked due to the "endogenous" value of data. With lemma 6, we can further explore the structure of binding constraints in the optimal selling mechanism. It can also help us tackle the tedious mutual incentive compatibility problem.

Define the single-valued correspondence $\gamma(\theta) : \Theta \to \Theta$ as below: (i) $\gamma(\theta) = \theta$ if $\theta = \lambda(\theta)$. (ii) $\gamma(\theta) \in \{a' \mid \text{IC}[\theta' \to \theta] \text{ is binding}\}$ if $\theta < \lambda(\theta)$. By lemma 6, $\{\theta' \mid \text{IC}[\theta' \to \theta] \text{ is binding}\}$ is a non-empty subset of $[\lambda(\theta), \overline{\theta}]$ if $a < \lambda(\theta)$. And if $\lambda(\theta) = \overline{\theta}$, $\gamma(\theta) = \overline{\theta}$.

We now explore the properties of $\lambda(\theta)$ and $\gamma(\theta)$.

Lemma 7 (Properties of $\lambda(\theta)$ and $\gamma(\theta)$)**.** *In the optimal menu,*

1. $\lambda(\theta) \le \lambda(\hat{\theta}) \le \gamma(\theta)$ *for* $\hat{\theta} \in [\theta, \lambda(\theta)]$
2. $\pi_2(\theta) : \Theta \to [0,1]$ *is non-increasing*
3. $\lambda(\theta) : \Theta \to \Theta$ *is non-decreasing*

A direct corollary from $\lambda(\hat{\theta}) \le \gamma(\theta)$ in statement 1 of lemma 7 is that $E_{\hat{\theta}}$ is not responsive for $\gamma(\theta)$. The three statement may seem standard in conventional mechanism design, however, with responsive IC, the analysis is highly complicated compared to the standard analysis.

With lemma 7, we now can characterize the mutual IC between θ and θ' given the order between θ' and $\lambda(\theta')$. Further, we can characterize derive a sharper prediction about the structure of the optimal mechanism.

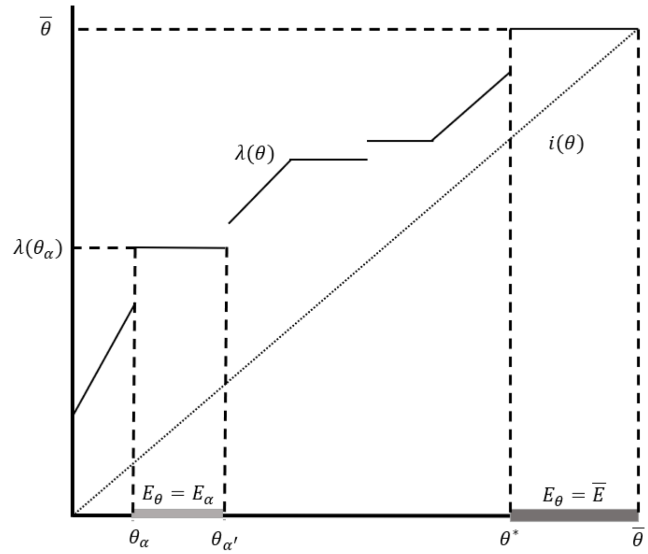

Figure 8: Tiered Pricing Mechanism

A mechanism is called tiered pricing mechanism if it implements the policy where the type space K is partitioned into intervals or singleton $\{I_\alpha\}_{\alpha\in\mathcal{A}}$, and in every partition set, the all types share the same menu, i.e. $(E_\theta, t_\theta) = (\mathrm{E}_\alpha, t_\alpha)$ for all $\theta \in I_\alpha$.

Lemma 8 (Structure of the Optimal Mechanism)**.** *The optimal mechanism is a tiered pricing mechanism whose partition $\mathcal{A}$ corresponds to the index set in the decomposition of $\lambda(\theta) = \sum_{\alpha\in\mathcal{A}} c_{\theta_\alpha} \mathbb{I}_{\theta\geq\theta_\alpha}$*

Step 3: Solving the Designer‘s Problem

With the structure of the optimal mechanism, we can now transform the constraints into some tractable forms for solving the designer‘s problem.

Denote $V(\theta) = V(E_\theta, \theta) - t_\theta$ as the net value of type θ.

Lemma 9 (Equivalent Transformation of Constraints)**.** *In the optimal mechanism, the* IC,IR and Responsiveness conditions are equivalent to

1. $\frac{1}{2}\pi_1(\theta) + t_\theta = t^*$ for all $\theta \in \Theta$, t^* is the associated tariff for all $\theta \in [\theta^*, \overline{\theta}]$
2. $V(\theta) = \int_0^\theta (1 - \pi_2(t))dt + V(\underline{\theta})$
3. IR[$\underline{\theta}$] holds
4. $\pi_2(\theta) : \Theta \to [0, 1]$ is non-increasing

A consequence of lemma 9 is that the designer‘s problem can be rewritten as a tractable form. Also notice that lemma 9 successfully separates the dimensions for design. And in the dimension of π_2, the optimization takes a similar form to standard one-dimensional screening.

Lemma 10 (Designer's Problem)**.** *The designer's problem of choosing the optimal π_2 is equivalent to solving the problem*

$$\max_{\pi_2(\theta)} \int_{\underline{\theta}}^{\overline{\theta}} \Phi(\theta) d\pi_2(\theta) \quad \text{s.t.} \quad \pi_2(\theta) : \Theta \to [0,1] \text{ is non-increasing}$$

where $\Phi(\theta) : \Theta \to \mathbb{R}$ *defined as* $\Phi(\theta) = \frac{1}{2\overline{\theta}} \left[\int_{\theta}^{\overline{\theta}} (1 - F(t) - tf(t)) dt \right] - m(\overline{\theta} - \theta)$.

Now the designer's problem consists of maximizing a linear functional subject to the constraint of its monotonicity. By the infinite-dimensional extension of Carathéodory's theorem [19], it follows that the optimal π_2 is an extreme point of the set of non-increasing allocation rules. By the characterization of the extreme points of the set of non-increasing allocation rules[20], we know that π_2 is a step function with $\operatorname{im} \pi_2 \subseteq \{0, 1\}$.

With the form of π_2 and the tiered-pricing structure, we can further deduce that the cardinality of tiers in the optimal mechanism. And then, we can transform the designer's problem as the choice of the optimal threshold θ^*. With following Lemma, we complete the proof of theorem 2.

Lemma 11 (Optimal Threshold)**.** $\theta^* \in \arg\max_\theta \theta \left(\overline{\theta} - \frac{1}{2} F(\theta) \right)$ *is the optimal threshold of the tiers.*

5 Future Work

We have solved the general case when the predictive power in one dimension is fixed, i.e. the design of multi-dimensional allocation and information under one-dimensional preference. The simplicity of the optimal menu relies on our simplifying assumption that the agent shares the same predictive power in some dimension. And the extension is extremely difficult. With one-dimensional preference (as in our case now), the varieties in the form of payoff function can be ranked in a monotone way and thus be pinned down finally. While in the multi-dimensional preference, the varieties may be messy, making it difficult to analyze the mutual IC.

Surprisingly, the structure of the optimal menu coincides with our work in the binary situation. Joint with the explorations of the structure of the constraints in the optimal menu, this result motivates us to try to tackle the design under multi-dimensional preference in our future work.

[19] See details in C.1.2

[20] See details in C.1.1

A Appendix: Proof of Lemma in Setup and Binary Case

A.1 Proof of Lemma 1

Consider any type n and experiment $E = (S, \pi)$. Without loss of generality, let every posterior of the type choose a single action after each signal. Let S^a denote the set of signals in experiment E that induces type n to choose action profile $a \in A^K$ for every posterior. Thus, $\cup_{a \in A^K} S^a = S$. Construct experiment $E' = (S', \pi')$ as a recommendation for type n based on experiment E, with signal space $S' = A^K$ and $\pi'(a|\omega) = \int_{S^a} \pi(s|\omega)$, for all $\omega \in \Omega$ and $a \in A^K$.

By construction, E' and E induce the same outcome distribution for type n; hence, $V(E', n) = V(E, n)$. Moreover, E' is a garbling of E. By Blackwell's theorem, we have $V(E', n') = V(E, n')$ for all n'. Therefore, for any incentive-compatible and individually rational direct mechanism $E(n), t(n)$, we can construct another direct mechanism $E'(n), t(n)$ whose experiments lead type n to take action a after observing recommendation profile s that is also incentive compatible and individually rational, thus yielding weakly larger profits.

A.2 Proof of Lemma 2

Suppose that there exists a responsive experiment for some type n implementing signal a, which recommends $a^{rj} = a_1$ and $a^{ri} = a_2$ for some pair (i, j) where $i > j$ in a responsive experiment for some type n, which requires that

$$\tfrac{\mu_{nj2}}{\mu_{nj1}} \leq \tfrac{\pi_{1k}}{\pi_{2k}} \leq \tfrac{\mu_{ni2}}{\mu_{ni1}}$$

Notice that the order the index of posteriors implies the order of likelihood-ratio between the two state, therefore $i > j$ implies

$$\tfrac{\mu_{ni2}}{\mu_{ni1}} \leq \tfrac{\mu_{nj2}}{\mu_{nj1}}$$

And the equality can not hold simultaneously, so we can derive the implementable set of recommendation profiles in lemma 2.

A.3 Proof of Lemma 3

If the fully informative experiment $\overline{\mathrm{E}}$ does not lie in the optimal menu, then choose the one charged the highest fee and replace the experiment with $\overline{\mathrm{E}}$, the revenue gets a weakly better improvement.

A.4 Proof of Lemma 4

Statement 1 is obvious because the two adjustments respectively recommend the same subset of posteriors of the type n to choose converse actions with the same probability in the same state, which induces the converse welfare changes.

The first outcome of statement 2 is to "restore" the valuation with fully informative experiment/without information.

Take the first adjustment as example, and the second adjustment is similar.

$$\Delta(a^{K+1} \xrightarrow{1} a^1 \mid E, n) \triangleq \frac{V(E(a^{K+1} \xrightarrow{i} a^1), n) - V(E, n)}{\delta} = \frac{(\sum_{k=1}^{K} \mu_{nk1} \Pr(\mu_{nk}) - 0)\delta}{\delta} = \sum_{k=1}^{K} \mu_{nk1} \Pr(\mu_{nk}) = \mu_1$$

The last equality comes from the splitting lemma.

We now prove the second outcome of statement 2

For all n, there exists k^* such that

$$\frac{\mu_{n11}}{\mu_{n12}} \geq \frac{\mu_{n21}}{\mu_{n22}} \geq \dots \geq \frac{\mu_{nk^*1}}{\mu_{nk^*2}} \geq 1 \geq \frac{\mu_{nk^*+11}}{\mu_{nk^*+12}} \geq \frac{\mu_{nK1}}{\mu_{nK2}}$$

Therefore

$$\begin{aligned}
&\Delta(a^{K+1} \xrightarrow{1} a^{k^*+1} \mid E, n) + \Delta(a^1 \xrightarrow{2} a^{k^*} \mid E, n) \\
=&\frac{V(E(a^{K+1} \xrightarrow{1} a^{k^*+1}), n) - V(E, n) + V(E(a^1 \xrightarrow{2} a^{k^*}), n) - V(E, n)}{\delta} \\
=&\sum_{k=1}^{k^*} \mu_{nk1} \Pr(\mu_{nk}) + \sum_{k=k^*+1}^{K} \mu_{nk2} \Pr(\mu_{nk}) \\
=&u_n
\end{aligned}$$

So we prove the second outcome.

A.5 Proof of Lemma 5

Proof of Statement (i) If there exists an experiment E for some type n in the optimal menu which is not non-dispersed, i.e. $\pi_{1K+1} \neq 0$ or $\pi_{21} \neq 0$. Suppose $\pi_{1K+1} \neq 0$ w.l.o.g.

Now consider the adjustment $\Delta(a^{K+1} \xrightarrow{1} a^1 \mid E, n)$ in this experiment. By lemma 4 it brings the net incremental value μ_1 to the type n while for other types, the net incremental value is weakly less than μ_1. Meanwhile, the obedience is also more relaxing.

Precisely speaking,

$$\frac{\pi_{11} + \pi_{1K+1}}{\pi_{21}} \geq \frac{\pi_{11}}{\pi_{21}} \geq \frac{\mu_{nK2}}{\mu_{nK1}}$$

$$\frac{0}{\pi_{2K+1}} \leq \frac{\pi_{1K+1}}{\pi_{2K+1}} \leq \frac{\mu_{n12}}{\mu_{n11}}$$

Therefore this adjustment allows the seller charge μ_1 more to the type n without violating IC, IR and Obedience constraints.

Proof of Statement (ii)

In the binary situation, the designer sells the fully informative experiment to one type (type I), and designs an experiment to another type (type II). By lemma 1, the experiment designed is responsive for type II.

Suppose it is not responsive for type I. There must exist some recommendation profile where the type I chooses another action profile after Bayesian updating when receiving them. By lemma 6, this experiment is non-dispersed. Therefore the signal (a_1, a_1) and (a_2, a_2) are responsive for type I. So the signal (a_1, a_2) is not responsive for type I.

Suppose (a_1, a_2) in fact recommends type I to choose (a_1, a_1) (the case (a_1, a_1) is similar to this one). Then considering the equivalent responsive form of this experiment and adjust π_{22} to $\pi 23$ until the signal (a_1, a_2) is responsive. And the designer can charge a strictly higher fee to type II without violating other constraints, which contracts the optimality of the menu.

A.6 Proof of Theorem 1

Now the designer‘s problem is:

$$\max_{E,T} \quad T$$

s.t.

$$\begin{aligned}
&T \leqslant \theta_1' \left(1-\pi_1\right) + \theta_2' \left(1-\pi_2\right) && \text{(IR-H)}\\
&T \leqslant \left(2\theta_1' - \theta_1\right)\left(1-\pi_1\right) + \left(2\theta_2' - \theta_2\right)\left(1-\pi_2\right) && \text{(IC-H)}\\
&\max\left\{\frac{\theta_2}{\frac{1}{2}-\theta_1} \frac{\theta_2'}{\frac{1}{2}-\theta_1'}\right\} = k_1 \leqslant \frac{\pi_1}{\pi_2} \leqslant k_2 = \min\left\{\frac{\frac{1}{2}-\theta_2}{\theta_1}, \frac{\frac{1}{2}-\theta_2'}{\theta_1'}\right\} && \text{(Responsiveness)}
\end{aligned}$$

And

$$\Delta = \text{RHS(IR-H)} - \text{RHS(IC-H)} = (\theta_1 - \theta_1')(1-\pi_1) + (\theta_2 - \theta_2')(1-\pi_2)$$

A.6.1 $\theta_1' \leq \theta_1$ and $\theta_2' \leq \theta_2$

In this case, $\Delta \geq 0$. So in the optimal menu, $(IC - H)$ is always binding. The designer‘s problem is now

$$\max \left(2\theta_1' - \theta_1\right)\left(1-\pi_1\right) + \left(2\theta_2' - \theta_2\right)\left(1-\pi_2\right)$$

s.t.

$$\max\left\{\frac{\theta_2}{\frac{1}{2}-\theta_1} \frac{\theta_2'}{\frac{1}{2}-\theta_1'}\right\} \leqslant \frac{\pi_1}{\pi_2} \leqslant \min\left\{\frac{\frac{1}{2}-\theta_2}{\theta_1}, \frac{\frac{1}{2}-\theta_2'}{\theta_1'}\right\} \quad \text{(Ob)}$$

Case 1: $2\theta_1' \geq \theta_1$ and $2\theta_2' \geq \theta_2$

Then all coefficients of π_1 and π_2 in the objective function is non-negative. So the optimal policy is $(\pi_1^*, \pi_2^*) = (1, 1)$

Case 2: $2\theta_1' \leq \theta_1$ and $2\theta_2' \leq \theta_2$

Then all coefficients of π_1 and π_2 in the objective function is non-positive. So the optimal policy is $(\pi_1^*, \pi_2^*) = (0, 0)$

Case 3: $2\theta_1' \geq \theta_1$ and $2\theta_2' \leq \theta_2$

Then the coefficient of π_1 is always non-positive. So in the optimal policy is $\frac{\pi_1}{\pi_2} = \max\left\{\frac{\theta_2}{\frac{1}{2}-\theta_1}\frac{\theta_2'}{\frac{1}{2}-\theta_1'}\right\}$

Notice that $\frac{1}{2} - \theta_1' \geq \frac{1}{2} - \theta_1 \geq (\frac{1}{2} - \theta_1)\frac{\theta_2'}{\theta_2}$ implies $\frac{\theta_2}{\frac{1}{2}-\theta_1} \geq \frac{\theta_2'}{\frac{1}{2}-\theta_1'}$

Therefore $\pi_1 = k_1\pi_2 = \frac{\theta_2}{\frac{1}{2}-\theta_1}\pi_2$ in the optimal menu

Now discuss the choice of optimal π_2

$$\max[-k_1 (2\theta_1' - \theta_1) - (2\theta_2' - \theta_2)]\pi_2 = (2\theta_1' - \theta_1) [\frac{2\theta_2'-\theta_2}{\theta_1-2\theta_1'} - k_1]\pi_2 \equiv (2\theta_1' - \theta_1) (k^* - k_1) \pi_2$$

s.t.

$$0 \leq \pi_2 \leq 1$$

Denote $F(\theta_1', \theta_2') = \theta_2(2\theta_1' - \theta_1) - (\theta_2 - 2\theta_2')(\frac{1}{2} - \theta_1)$

Notice that $F(\frac{\theta_1}{2}, \frac{\theta_2}{2}) = 0$, $F(\frac{\theta_1}{2}, 0) = -\theta_2(\frac{1}{2} - \theta_1) < 0$ and $F(\theta_1, \frac{\theta_2}{2}) = \theta_1\theta_2 > 0$ and $F(\theta_1, 0) = 2\theta_2(\theta_2 - \frac{1}{4})$

So in this zone, the (θ_1', θ_2') is sold $(\pi_1^*, \pi_2^*) = (0, 0)$ when $F(\theta_1', \theta_2') > 0$ while sold $(\pi_1^*, \pi_2^*) = (k_1, 1)$ when $F(\theta_1', \theta_2') \leq 0$, and both situations always exist.

Case 4: $2\theta_1' \leq \theta_1$ and $2\theta_2' \geq \theta_2$

This case is similar to case 1.3 and in this zone, the (θ_1', θ_2') is sold either $(\pi_1^*, \pi_2^*) = (0, 0)$ or $(\pi_1^*, \pi_2^*) = (1, \frac{1}{k_2})$, and both situations always exist.

Notice that $\theta_1 + \theta_2 \leq \frac{1}{2}$ so at least one of $(\theta_1, 0)$ and $(0, \theta_2)$ is sold a partially informative experiment $((\pi_1^*, \pi_2^*) = (k_1, 1)$ or $(\pi_1^*, \pi_2^*) = (1, \frac{1}{k_2})$)

A.6.2 $\theta_1 \leq \theta_1'$ and $\theta_2 \geq \theta_2'$

Similar to the discussions in case 1.3, in the optimal policy, $\frac{\pi_1}{\pi_2} = k_1$, i.e. $\pi_1 = k_2\pi_2 = \frac{\theta_2}{\frac{1}{2}-\theta_1}\pi_2$

$$\max T$$

s.t.

$$
\begin{aligned}
&T \leqslant (\theta_1' + \theta_2') - (k_1\theta_1' + \theta_2')\,\pi_2 && (IR-H)\\
&T \leqslant (2\theta_1' + 2\theta_2' - \theta_1 - \theta_2) + [(2\theta_2' - \theta_2) - k_1\,(2\theta_1' - \theta_1)]\pi_2 && (IC-H)
\end{aligned}
$$

Recall that $F(\theta_1', \theta_2') = \theta_2(2\theta_1' - \theta_1) - (\theta_2 - 2\theta_2')(\frac{1}{2} - \theta_1)$. So in this zone, the (θ_1', θ_2') is sold $(\pi_1^*, \pi_2^*) = (0, 0)$ when $F(\theta_1', \theta_2') > 0$

When $F(\theta_1', \theta_2') \leq 0$, notice that $\Delta = (IR-H) - (IC-H) = (\theta_1' - \theta_1)\pi_1 + (\theta_2' - \theta_2)\pi_2 + (\theta_1 + \theta_2 - \theta_1' - \theta_2') = [(\theta_2' - \theta_2) + k_1(\theta_1' - \theta_1)]\pi_2 + (\theta_1 + \theta_2 - \theta_1' - \theta_2')$ is positive when $\pi_2 = 0$ while negative when $\pi_2 = 1$, by the continuity and linearity, the optimal π_2^* is the interior point in $[0, 1]$, i.e.

$$
\begin{aligned}
&[(\theta_2' - \theta_2) + k_1(\theta_1' - \theta_1)]\pi_2 + (\theta_1 + \theta_2 - \theta_1' - \theta_2') = 0\\
&\Rightarrow \pi_2^* = \tfrac{\theta_1 - \theta_1' + \theta_2 - \theta_2'}{\theta_2 - \theta_2' + k_1(\theta_1 - \theta_1')} \text{ and } \pi_1^* = k_1\pi_2^*
\end{aligned}
$$

So the optimal policy is $(\pi_1^*, \pi_2^*) = (k_1\frac{\theta_1 - \theta_1' + \theta_2 - \theta_2'}{\theta_2 - \theta_2' + k_1(\theta_1 - \theta_1')}, \frac{\theta_1 - \theta_1' + \theta_2 - \theta_2'}{\theta_2 - \theta_2' + k_1(\theta_1 - \theta_1')})$ when $F(\theta_1', \theta_2') \leq 0$, and $(\pi_1^*, \pi_2^*) = (0, 0)$ otherwise.

A.6.3 $\theta_1 \geq \theta_1'$ and $\theta_2 \leq \theta_2'$

Case 3 is similar to case 2 and we omit here.

B Appendix: Proof of Lemma in the General Case

B.1 Proof of the Optimal Responsiveness Zone

If $\frac{\pi_1(\theta)}{\pi_2(\theta)} > 1$ for some θ, then for other θ' E_θ either recommends the same action profile with the same probability

E_θ for θ'	(a_1, a_1)	(a_1, a_2)	(a_2, a_2)
ω_1	$1 - \pi_2$	π_2	0
ω_2	0	π_1	$1 - \pi_1$

or pools the recommendation (a_1, a_2) with (a_2, a_2)

E_θ for θ'	(a_1, a_1)	(a_2, a_2)
ω_1	$1 - \pi_2$	π_2
ω_2	0	1

In the first case, the payoff for θ' is $V(E_\theta, \theta') = \theta' + m - m\pi_1 - \theta'\pi_2$. In the second case, the payoff is $V(E_\theta, \theta') = \theta' + m - \frac{1}{2}\pi_2$. Notice that adjust π_1 to $1 - \pi_1$ will increase the fees charging to a without violating the IC conditions. We can adjust in this way until all $\frac{\theta}{\frac{1}{2} - m} \leq \frac{\pi_1(\theta)}{\pi_2(\theta)} \leq 1$.

B.2 Proof of Lemma 6

Proof of Statement 1

We first prove that if $E_{\theta^*} = \overline{E}$ for some $\theta^* \in \Theta$, then for all $\theta > \theta^*$, $E_\theta = \overline{E}$.

Suppose there exists $\theta > \theta^*$, $E_\theta \neq \overline{E}$.

Then considering the IC[$\theta \to \theta^*$] and IC[$\theta^* \to \theta$]

IC[$\theta \to \theta^*$]

$$\theta + m - m\pi_1(\theta) - \theta\pi_2(\theta) - t_\theta \geq \theta + m - t_{\theta^*}$$
$$-m\pi_1(\theta) - \theta\pi_2(\theta) \geq t_\theta - t_{\theta^*}$$

IC[$\theta^* \to \theta$]

$$\theta^* + m - t_{\theta^*} \geq \theta^* + m - m\pi_1(\theta) - \theta^*\pi_2(\theta) - t_\theta$$
$$t_\theta - t_{\theta^*} \geq -m\pi_1(\theta) - \theta^*\pi_2(\theta)$$

Combining the two equations and we get $-\theta\pi_2(\theta) \geq -\theta^*\pi_2(\theta)$, which implies that $\pi_2(\theta) = 0$ since $\theta > \theta^*$. Thus, $E_\theta = \overline{E}$, a contradiction.

By the existence of the fully informative experiment, the set $I_{\overline{E}}$ defined as $\{\theta | E_\theta = \overline{E}\}$ is non-empty. Define $\theta^* = \inf I_{\overline{E}}$. We can derive the closeness of $I_{\overline{E}}$, , which means $\theta^* = \min I_{\overline{E}}$, by verifying the optimality of the menu and we complete the proof of closeness after proving statement 2.

Proof of Statement 2

We first prove that $\theta < \lambda(\theta)$ for all $\theta < \theta^*$.

Suppose there exist $\theta < \theta^*$ whose responsiveness constraints are binding, i.e $\theta = \lambda(\theta) = \overline{\theta}\frac{\pi_1(\theta)}{\pi_2(\theta)}$.

Therefore, $V(E_\theta, \theta) = \theta + m - \theta\pi_2(\theta) - m\pi_1(\theta) = \theta + m - \frac{1}{2}\pi_1(\theta)$ and $V(E_{\theta'}, \theta') = \theta' + m - \frac{1}{2}\pi_1(\theta')$ for any $\theta' \in (\theta, \theta^*)$.

$$\begin{array}{ccc}
 & V(E_\theta, \theta) - t_\theta & \\
\geq & V(E_{\theta'}, \theta) - t_{\theta'} & (\text{IC}[\theta \to \theta']) \\
= & u(E_{\theta'}, \theta) - U(\theta) - t_{\theta'} & \\
\geq & u(E_{\theta'}, \theta') - U(\theta') - t_{\theta'} + U(\theta') - U(\theta) & (\text{monotonicity in } \theta \text{ of } u(E,\theta)) \\
= & V(E_{\theta'}, \theta') - t_{\theta'} + U(\theta') - U(\theta) & \\
\geq & V(E_\theta, \theta') - t_\theta + U(\theta') - U(\theta) & (\text{IC}[\theta' \to \theta]) \\
= & V(E_\theta, \theta) - t_\theta & (\text{binding responsiveness})
\end{array}$$

The first inequality is from IC[$\theta \to \theta'$]. The second inequality is from the (negative) single-crossing property between π_2 and θ of $u(E, \theta) = 1 - \theta\pi_2 - m\pi_1$. And the equality is attained if and only if $\pi_2 = 0$. The third inequality is from IC$[\theta' \to \theta]$. And the last equality is from the binding responsiveness of E_θ for θ. We conclude that all equality conditions above should be satisfied. Therefore, $\pi_2(\theta') = 0$ and by the monotonicity of $u(E,\theta)$. So $E_{\theta'} = \overline{E}$, which contradicts that $\theta' < \theta^*$.

Now we prove the existence of binding (non-responsive) IC.

Motivated by the proof of $\frac{\theta}{\frac{1}{2}-m} \leq \frac{\pi_1(\theta)}{\pi_2(\theta)} \leq 1$, we can keep adjusting π_1 to $1-\pi_1$. By $\frac{\theta}{\frac{1}{2}-m} \leq \frac{\pi_1(\theta)}{\pi_2(\theta)} \leq 1$ for all θ. Therefore for other θ', E_θ either recommends the same action profile with the same probability

E_θ for θ'	(a_1, a_1)	(a_1, a_2)	(a_2, a_2)
ω_1	$1-\pi_2$	π_2	0
ω_2	0	π_1	$1-\pi_1$

or pools the recommendation (a_1, a_2) with (a_1, a_1)

E_θ for θ'	(a_1, a_1)	(a_2, a_2)
ω_1	1	0
ω_2	π_1	$1-\pi_1$

In the first case, the payoff for θ' is $V(E_\theta, \theta') = \theta' + m - m\pi_1 - \theta'\pi_2$. In the second case, the payoff is $V(E_\theta, \theta') = \theta' + m - \frac{1}{2}\pi_1$. Notice that adjustment from π_1 to $1-\pi_1$ is profitable to the designer without violating other conditions until either the responsiveness is binding, or for some θ' who is recommended a pooling recommendation profile by E_θ, the IC $[\theta' \to \theta]$ is binding. By the proof above, the responsiveness is never binding for the $\theta < \theta^*$. Therefore, we can derive the statement 2.

With statement 2, we can further complete the proof of closeness in statement 1, which means $\theta^* = \min I_{\overline{E}}$. If $\lambda(\theta^*) > \theta^*$, i.e. the responsiveness of θ^* is not binding, then there exist $\theta' > \theta^*$, IC $[\theta' \to \theta]$ is binding, which means that $t_{\theta^*} + \frac{1}{2}\pi_1(\theta^*) = t^*$, where t^* is the associated tariff of those sold the fully informative one. For some $\theta^* < \hat{\theta} < \lambda(\theta^*)$, IC$[\hat{\theta} \to \theta^*]$ implies that $-t^* \geq -\hat{\theta}\pi_2(\theta^*) - m\pi_1(\theta^*) - t_{\theta^*}$, i.e. $t_{\theta^*} + \frac{1}{2}\pi_1(\theta^*) \leq \hat{\theta}\pi_2(\theta^*) + m\pi_1(\theta^*) + t_{\theta^*}$. Then we have $\lambda(\theta^*) \leq \hat{\theta}$, which is contradictory.

If $\lambda(\theta^*) = \theta^*$, for $\theta' > \theta^*$, IC $[\theta' \to \theta^*]$ implies that $-t^* \geq -\frac{1}{2}\pi_1(\theta^*) - t_{\theta^*}$, while IC $[\theta^* \to \theta']$ implies that $-\frac{1}{2}\pi_1(\theta^*) - t_{\theta^*} \geq -t^*$, which means that $t_{\theta^*} + \frac{1}{2}\pi_1(\theta^*) = t^*$. Therefore θ^* is indifferent between the menu of her own and those of θ where $\theta > \theta^*$. The designer can strictly increase her revenue by replacing the menu of θ^* to the fully informative one, because $t_{\theta^*} < t^*$, without violating other conditions.

B.3 Proof of Lemma 7

If $\theta \in \left[\theta^*, \overline{\theta}\right]$, the conclusion trivially holds.

Now we discuss that $\theta \in [\underline{\theta}, \theta^*)$ where responsiveness is not binding a.e, i.e. $\lambda(\theta) > \theta$.

In the optimal menu, for $\hat{\theta} \in [\theta, \lambda(\theta)]$. E_θ is responsive for $\hat{\theta}$. Since $\hat{\theta} > \theta$, $E_{\hat{\theta}}$ is also responsive for θ.

We first prove that $\pi_2(\theta) \geq \pi_2(\hat{\theta})$

IC$[\theta \to \hat{\theta}]$

$$\begin{aligned}
V(E_\theta, \theta) - t_\theta &\geq V(E_{\hat{\theta}}, \theta) - t_{\hat{\theta}} \\
\theta + m - a\pi_2(\theta) - m\pi_1(\theta) - t_\theta &\geq \theta + m - \theta\pi_2(\hat{\theta}) - m\pi_1(\hat{\theta}) - t_{\hat{\theta}} \\
t_{\hat{\theta}} - t_\theta &\geq \theta[\pi_2(\theta) - \pi_2(\hat{\theta})] + m[\pi_1(\theta) - \pi_1(\hat{\theta})]
\end{aligned}$$

IC$[\hat{\theta} \to \theta]$

$$\begin{aligned}
V(E_{\hat{\theta}}, \hat{\theta}) - t_{\hat{\theta}} &\geq V(E_\theta, \hat{\theta}) - t_\theta \\
\hat{\theta} + m - \hat{\theta}\pi_2(\hat{\theta}) - m\pi_1(\hat{\theta}) - t_{\hat{\theta}} &\geq \hat{\theta} + m - \hat{\theta}\pi_2(\theta) - m\pi_1(\theta) - t_\theta \\
\hat{\theta}[\pi_2(\theta) - \pi_2(\hat{\theta})] + m[\pi_1(\theta) - \pi_1(\hat{\theta})] &\geq t_{\hat{\theta}} - t_\theta
\end{aligned}$$

By IC$[\hat{\theta} \to \theta]$ and IC$[\hat{\theta} \to \hat{\theta}]$, we derive that

$$(\hat{\theta} - \theta)[\pi_2(\theta) - \pi_2(\hat{\theta})] \geq 0$$

Therefore $\pi_2(\theta) \geq \pi_2(\hat{\theta})$

B.3.1 Proof of Statement 1

We now prove the statement when $\pi_2(\theta) \geq \pi_2(\hat{\theta}) > 0$.

Part 1 $\lambda(\hat{\theta}) \leq \gamma(\theta)$

If $\lambda(\hat{\theta}) > \gamma(\theta) > \hat{\theta}$, then $E_{\hat{\theta}}$ is responsive for $\gamma(\theta)$, we have

$$\gamma(\hat{\theta}) > \lambda(\hat{\theta}) > \gamma(\theta) \geq \lambda(\theta) \geq \hat{\theta} > \theta$$

Therefore, both the responsiveness of θ and $\hat{\theta}$ are not binding while the IC$[\gamma(\theta) \to \theta]$ and IC$[\gamma(\hat{\theta}) \to \hat{\theta}]$ are binding.

$$\begin{aligned}
V(E_\theta, \gamma(\theta)) - t_\theta &\geq V(E_{\hat{\theta}}, \gamma(\theta)) - t_{\hat{\theta}} \\
\gamma(\theta) + m - \tfrac{1}{2}\pi_1(\theta) - t_\theta &\geq \gamma(\theta) + m - \gamma(\theta)\,\pi_2(\hat{\theta}) - m\pi_1(\hat{\theta}) - t_{\hat{\theta}} \\
t_{\hat{\theta}} - t_\theta &\geq \tfrac{1}{2}\pi_1(\theta) - \gamma(\theta)\,\pi_2(\hat{\theta}) - m\pi_1(\hat{\theta})
\end{aligned}$$

IC$[\gamma(\hat{\theta}) \to \hat{\theta}]$ is binding, while IC$[\gamma(\hat{\theta}) \to \theta]$ may not be binding. (Moreover, E_θ is also not responsive for $\gamma(\hat{\theta})$.)

$$\begin{gathered} V(E_{\hat{\theta}}, \gamma(\hat{\theta})) - t_{\hat{\theta}} \geq V(E_\theta, \gamma(\hat{\theta})) - t_\theta \\ \gamma(\hat{\theta}) + m - \tfrac{1}{2}\pi_1(\hat{\theta}) - t_{\hat{\theta}} \geq \gamma(\hat{\theta}) + m - \tfrac{1}{2}\pi_1(\theta) - t_\theta \\ \tfrac{1}{2}\pi_1(\theta) - \tfrac{1}{2}\pi_1(\hat{\theta}) \geq t_{\hat{\theta}} - t_\theta \end{gathered}$$

Combining the two equations above, we have

$$\begin{gathered} \tfrac{1}{2}\pi_1(\theta) - \tfrac{1}{2}\pi_1(\hat{\theta}) \geq \tfrac{1}{2}\pi_1(\theta) - \gamma(\theta)\,\pi_2(\hat{\theta}) - m\pi_1(\hat{\theta}) \\ \gamma(\theta)\,\pi_2(\hat{\theta}) - \overline{\theta}\pi_1(\hat{\theta}) \geq 0 \\ \gamma(\theta) \geq \gamma(\hat{\theta}) \end{gathered}$$

which contradicts that $\gamma(\theta) < \gamma(\hat{\theta})$.

Therefore $\lambda(\hat{\theta}) \in [\hat{\theta}, \gamma(\theta)]$, or $E_{\hat{\theta}}$ is not responsive for $\lambda(\theta)$

Part 2 $\lambda(\hat{\theta}) \geq \lambda(\theta)$

If there exists $\theta < \hat{\theta} \leq \lambda(\theta)$, $\lambda(\hat{\theta}) < \lambda(\theta)$.

IC$[\gamma(\theta) \to \theta]$ is binding while IC$[\gamma(\theta) \to \hat{\theta}]$ may not be binding. Moreover, $E_{\hat{\theta}}$ is not responsive for $\gamma(\theta)$ because $\lambda(\hat{\theta}) < \gamma(\theta)$

$$\begin{gathered} V(E_\theta, \gamma(\theta)) - t_\theta \geq V(E_{\hat{\theta}}, \gamma(\theta)) - t_{\hat{\theta}} \\ \gamma(\theta) + m - \tfrac{1}{2}\pi_1(\theta) - t_\theta \geq \gamma(\theta) + m - \tfrac{1}{2}\pi_1(\hat{\theta}) - t_{\hat{\theta}} \\ t_{\hat{\theta}} - t_\theta \geq \tfrac{1}{2}\pi_1(\theta) - \tfrac{1}{2}\pi_1(\hat{\theta}) \end{gathered}$$

IC$[\hat{\theta} \to \theta]$

$$\hat{\theta}[\pi_2(\theta) - \pi_2(\hat{\theta})] + m[\pi_1(\theta) - \pi_1(\hat{\theta})] \geq t_{\hat{\theta}} - t_\theta$$

Combining the two inequalities, we have

$$\begin{gathered} \hat{\theta}[\pi_2(\theta) - \pi_2(\hat{\theta})] + m[\pi_1(\theta) - \pi_1(\hat{\theta})] \geq \tfrac{1}{2}\pi_1(\theta) - \tfrac{1}{2}\pi_1(\hat{\theta}) \\ \hat{\theta}\left[\pi_2(\theta) - \pi_2(\hat{\theta})\right] \geq \overline{\theta}\left[\pi_1(\theta) - \pi_1(\hat{\theta})\right] \\ \hat{\theta}[\pi_2(\theta) - \pi_2(\hat{\theta})] \geq \lambda(\theta)\,\pi_2(\theta) - \lambda(\hat{\theta})\pi_2(\hat{\theta}) \\ [\hat{\theta} - \lambda(\hat{\theta})][\pi_2(\theta) - \pi_2(\hat{\theta})] + [\lambda(\hat{\theta}) - \lambda(\theta)]\pi_2(\theta) \geq 0 \end{gathered}$$

By $\hat{\theta} < \lambda(\hat{\theta})$ and $\lambda(\hat{\theta}) < \lambda(\theta)$, we have

$$0 > [\hat{\theta} - \lambda(\hat{\theta})][\pi_2(\theta) - \pi_2(\hat{\theta})] + [\lambda(\hat{\theta}) - \lambda(\theta)]\pi_2(\theta) \geq 0$$

which is impossible.

Therefore, $\lambda(\hat{\theta}) \geq \lambda(\theta)$ for $\hat{\theta} \in [\theta, \lambda(\theta)]$

This completes the proof of statement 1.

B.3.2 Proof of Statement 2: Monotonicity on $[\theta, \lambda(\hat{\theta})]$

Now we first prove the $\pi_2(\hat{\theta})$ is non-increasing on $\hat{\theta} \in [\theta, \lambda(\theta)]$

Considering $\hat{\theta}' > \hat{\theta}$. By $\lambda(\theta) \leq \min\{\lambda(\hat{\theta}'), \lambda(\hat{\theta})\}$, $E_{\hat{\theta}'}$ and $E_{\hat{\theta}}$ are both responsive for $\hat{\theta}'$ and $\hat{\theta}$.

IC$[\hat{\theta}' \to \hat{\theta}]$

$$\begin{gathered}
V(E_{\hat{\theta}'}, \hat{\theta}') - t_{\hat{\theta}'} \geq V(E_{\hat{\theta}}, \hat{\theta}) - t_{\hat{\theta}} \\
\hat{\theta}' + m - \hat{\theta}'\pi_2(\hat{\theta}') - m\pi_1(\hat{a}') - t_{\hat{\theta}'} \geq \hat{\theta}' + m - \hat{\theta}'\pi_2(\hat{\theta}) - m\pi_1(\hat{\theta}) - t_{\hat{\theta}} \\
t_{\hat{\theta}} - t_{\hat{\theta}'} \geq \hat{\theta}'[\pi_2(\hat{\theta}') - \pi_2(\hat{\theta})] + m[\pi_1(\hat{\theta}') - \pi_1(\hat{\theta})]
\end{gathered}$$

IC$[\hat{\theta} \to \hat{\theta}']$

$$\begin{gathered}
V(E_{\hat{\theta}}, \hat{\theta}) - t_{\hat{\theta}} \geq V(E_{\hat{\theta}'}, \hat{\theta}) - t_{\hat{\theta}'} \\
\hat{\theta} + m - \hat{\theta}\pi_2(\hat{\theta}) - m\pi_1(\hat{\theta}) - t_{\hat{a}} \geq \hat{\theta} + m - \hat{\theta}\pi_2(\hat{\theta}') - m\pi_1(\hat{\theta}') - t_{\hat{\theta}'} \\
\hat{\theta}[\pi_2(\hat{\theta}') - \pi_2(\hat{\theta})] + m[\pi_1(\hat{\theta}') - \pi_1(\hat{\theta})] \geq t_{\hat{\theta}} - t_{\hat{\theta}'}
\end{gathered}$$

By IC$[\hat{\theta} \to \hat{\theta}']$ and IC$[\hat{\theta}' \to \hat{\theta}]$, we derive that

$$(\hat{\theta} - \hat{\theta}')[\pi_2(\hat{\theta}') - \pi_2(\hat{\theta})] \geq 0$$

Therefore $\pi_2(\hat{\theta}) \geq \pi_2(\hat{\theta}')$.

B.3.3 Proof of Statement 3: Monotonicity on $[\theta, \lambda(\hat{\theta})]$

Considering $\theta < \hat{\theta} < \hat{\theta}' < \lambda(\theta)$, by the result of **Part 2** in the **Proof of Statement 1**, we learn that $\hat{\theta} < \hat{\theta}' < \lambda(\hat{\theta})$. Reuse that result on $\hat{\theta}$ to get $\lambda(\hat{\theta}) < \lambda(\hat{\theta}')$.

B.3.4 Proof of Statement 2 and 3: Global Monotonicity

If there exists $\theta \neq \theta^*$, $\lambda(\lambda(\theta)) = \lambda(\theta)$

By the monotonicity of $\lambda(\hat{\theta})$, $\lambda(\hat{\theta}) = \lambda(\theta)$ for all $\hat{\theta} \in [\theta, \lambda(\theta)]$

We now prove that $\lambda(\theta) = \theta^*$

If $\lambda(\theta) < \theta^*$, for any $\lambda(\theta) < \theta' < \theta^*$

IC$[\theta' \to \lambda(\theta)]$

$$\theta' + m - \theta'\pi_2(\theta') - m\pi_1(\theta') - t_{\theta'} \geq \theta' + m - \tfrac{1}{2}\pi_1\left(\lambda(\theta)\right) - t_{\lambda(\theta)}$$
$$\tfrac{1}{2}\pi_1\left(\lambda(\theta)\right) - \theta'\pi_2(\theta') - m\pi_1(\theta') \geq t_{\theta'} - t_{\lambda(\theta)}$$

IC$[\lambda(\theta) \to \theta']$

$$\lambda(\theta) + m - \lambda(\theta)\pi_2\left(\lambda(\theta)\right) - m\pi_1\left(\lambda(\theta)\right) - t_{\lambda(\theta)} \geq \lambda(\theta) + m - \lambda(\theta)\pi_2(\theta') - m\pi_1(\theta') - t_{\theta'}$$
$$t_{\theta'} - t_{\lambda(\theta)} \geq \lambda(\theta)\pi_2\left(\lambda(\theta)\right) + m\pi_1\left(\lambda(\theta)\right) - \lambda(\theta)\pi_2(\theta') - m\pi_1(\theta')$$

Combining the two equations and we have

$$\tfrac{1}{2}\pi_1\left(\lambda(\theta)\right) - \theta'\pi_2(\theta') - m\pi_1(\theta') \geq \lambda(\theta)\pi_2\left(\lambda(\theta)\right) + m\pi_1\left(\lambda(\theta)\right) - \lambda(\theta)\pi_2(\theta') - m\pi_1(\theta')$$
$$(\lambda(\theta) - \theta')\pi_2(\theta') \geq 0$$

which is contradictory.

This implies that $\lambda(\theta) = \lambda(\lambda(\theta))$ can only occur when $\lambda(\theta) = \theta^*$. And in this case, the $\lambda(\theta) = \theta^* < \overline{\theta} = \lambda(\theta^*)$, and $\pi_2(\theta) > 0 = \pi_2(\theta^*)$.

For all θ where $\lambda(\theta) < \lambda(\lambda(\theta))$, there always exist $\hat{\theta} \in [\theta, \lambda(\theta)]$, $\lambda(\hat{\theta}) > \lambda(\theta)$, so the monotonicity can be transitive across different $[\theta, \lambda(\theta)]$[21].

Above all, the monotonicity of $\lambda(\theta)$ and $\pi_2(\theta)$ can always be transitive across different $[\theta, \lambda(\theta)]$, thus proving the global monotonicity.

B.4 Proof of Lemma 8

It is equivalent to show that, if $\lambda(\theta) = \lambda(\theta')$ for some $\theta' > \theta$, then for all $\hat{\theta} \in [\theta, \theta']$, $\hat{\theta}$ share the same menu with θ, i.e. $(\pi_1(\hat{\theta}), \pi_2(\hat{\theta}), t_{\hat{\theta}}) = (\pi_1(\theta), \pi_2(\theta), t_\theta)$.

By $\lambda(\theta) = \lambda(\theta')$ and $\lambda(\hat{\theta})$ is non-decreasing for $\hat{\theta} \in [\theta, \lambda(\theta)]$ by lemma 7, $\lambda(\hat{\theta}) = \lambda(\theta) = \lambda(\theta') < \gamma(\theta)$ for $\hat{\theta} \in [\theta, \theta']$. We know that both E_θ and $E_{\hat{\theta}}$ are responsive for both θ and $\hat{\theta}$, and both not responsive for $\gamma(\theta)$.

IC$[\gamma(\theta) \to \theta]$ is binding while IC$[\gamma(\theta) \to \hat{\theta}]$ may not be binding.

$$V(E_\theta, \gamma(\theta)) - t_\theta \geq V(E_{\hat{\theta}}, \gamma(\theta)) - t_{\hat{\theta}}$$
$$t_{\hat{\theta}} - t_\theta \geq \tfrac{1}{2}\pi_1(\theta) - \tfrac{1}{2}\pi_1(\hat{\theta})$$

[21] To guarantee the validity of the transitivity (of monotonicity) across different $[\theta, \lambda(\theta))$, it remains to exclude one situation where for all $\hat{\theta} \in [\theta, \lambda(\theta))$, $\lambda(\hat{\theta}) = \lambda(\theta)$ when $\lambda(\theta) \neq \lambda(\lambda(\theta))$. See in C.2

IC[$\hat{\theta} \to \theta$]

$$V(E_{\hat{\theta}}, \hat{\theta}) - t_{\hat{\theta}} \geq V(E_\theta, \hat{\theta}) - t_\theta$$
$$\hat{\theta}[\pi_2(\theta) - \pi_2(\hat{\theta})] + m[\pi_1(\theta) - \pi_1(\hat{\theta})] \geq t_{\hat{\theta}} - t_\theta$$

Combining the two inequalities, we have

$$[\hat{\theta} - \lambda(\hat{\theta})][\pi_2(\theta) - \pi_2(\hat{\theta})] + [\lambda(\hat{\theta}) - \lambda(\theta)]\pi_2(\theta) \geq 0$$
$$[\hat{\theta} - \lambda(\hat{\theta})][\pi_2(\theta) - \pi_2(\hat{\theta})] \geq 0$$

So we have $\pi_2(\hat{\theta}) = \pi_2(\theta)$ for all $\hat{\theta}$. By $\lambda(\hat{\theta}) = \lambda(\theta)$, we have $\pi_1(\hat{\theta}) = \pi_1(\theta)$. Revisiting the IC conditions between $\hat{\theta}$ and θ, we can further derive that $t_{\hat{\theta}} = t_\theta$.

B.5 Proof of Lemma 9

B.5.1 Necessity

With IC, IR and Responsiveness,

Statement 1: We first prove that for all $\hat{\theta} \in [\theta, \lambda(\theta))$,

$$\tfrac{1}{2}\pi_1(\theta) - \tfrac{1}{2}\pi_1(\hat{\theta}) = t_{\hat{\theta}} - t_\theta$$

By $\gamma(\hat{\theta}) \geq \lambda(\hat{\theta}) > \lambda(\theta) > \hat{\theta}$, IC[$\gamma(\hat{\theta}) \to \theta$] is binding, while IC[$\gamma(\hat{\theta}) \to \hat{\theta}$] may not be binding

$$V(E_{\hat{\theta}}, \gamma\left(\hat{\theta}\right)) - t_{\hat{\theta}} \geq V(E_\theta, \gamma\left(\hat{\theta}\right)) - t_\theta$$
$$\gamma\left(\hat{\theta}\right) + m - \tfrac{1}{2}\pi_1(\hat{\theta}) - t_{\hat{\theta}} \geq \gamma\left(\hat{\theta}\right) + m - \tfrac{1}{2}\pi_1(\theta) - t_\theta$$
$$\tfrac{1}{2}\pi_1(\theta) - \tfrac{1}{2}\pi_1(\hat{\theta}) \geq t_{\hat{\theta}} - t_\theta$$

With IC[$\gamma(\theta) \to \theta$] is binding, while IC[$\gamma(\theta) \to \hat{\theta}$] may not be binding, we have

$$\tfrac{1}{2}\pi_1(\theta) - \tfrac{1}{2}\pi_1(\hat{\theta}) \leq t_{\hat{\theta}} - t_\theta$$

Therefore, for all $\hat{\theta} \in [\theta, \lambda(\theta))$,

$$\tfrac{1}{2}\pi_1(\theta) - \tfrac{1}{2}\pi_1(\hat{\theta}) = t_{\hat{\theta}} - t_\theta$$

Now we prove that for all $\theta, \hat{\theta} \in \Theta$, $\theta < \hat{\theta}$.

$$\frac{1}{2}\pi_1(\theta) - \frac{1}{2}\pi_1(\hat{\theta}) = t_{\hat{\theta}} - t_\theta$$

By the proof above, we can also get that for any $\theta \leq \hat{\theta}' < \lambda(\theta)$,

$$\frac{1}{2}\pi_1(\theta) - \frac{1}{2}\pi_1(\hat{\theta}') = t_{\hat{\theta}'} - t_\theta$$

Therefore, for all $\theta \leq \hat{\theta} < \hat{\theta}' < \lambda(\theta)$,

$$\frac{1}{2}\pi_1(\hat{\theta}') - \frac{1}{2}\pi_1(\hat{\theta}) = t_{\hat{\theta}} - t_{\hat{\theta}'}$$

By that $\lambda(\theta) > \theta$ a.e. in Θ, this relation can be transitive across different $[\theta, \lambda(\theta))$. Therefore, for all $\theta \in \Theta$, $\frac{1}{2}\pi_1(\theta) + t_\theta = t^*$.

Statement 2: With statement 1, we can reduce the net value function into one dimension.

$$\begin{aligned} V(\theta) &= \theta + m - \theta\pi_2(\theta) - m\pi_1(\theta) - t_\theta \\ &= \theta + m - \theta\pi_2(\theta) - 2m(t^* - t_\theta) - t_\theta \\ &= \theta(1 - \pi_2(\theta)) - 2\overline{\theta}t_\theta + m(1 - 2t^*) \end{aligned}$$

By the optimal structure of the menu, for any $\theta \in [\underline{\theta}, \theta^*)$, there always exist ϵ, $E_{\theta'}$ is always responsive for θ, $\theta' \in (\theta - \epsilon, \theta + \epsilon)$. $V(\pi_2, \theta)$ is differentiable and absolutely continuous for on $\theta \in [\underline{\theta}, \theta^*)$[22]. By the envelope theorem (Milgrom and Segal (2002), Sinander (2022)), for all such θ,

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} V_t(\pi_2, t)dt = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta}(1 - \pi_2(t))dt$$

Statement 3 and 4: Statement 3 trivialy holds when all IR conditions hold. Statement 4 is from lemma 7.

B.5.2 Sufficiency

Incentive Compatibility

Construct $t_\theta = \frac{\theta}{2\overline{\theta}}(1 - \pi_2(\theta)) + t_{\underline{\theta}} - \frac{\int_{\underline{\theta}}^{\theta}(1-\pi_2(t))dt}{2\overline{\theta}}$.

i.Responsive IC

IC$[\theta \to \theta']$ where $E_{\theta'}$ is responsive for θ

[22] Here only for θ^*, the differentiability of $V(\theta)$ may not hold when $\lambda(\theta) = \theta^*$ for $\theta \in (\theta - \epsilon, \theta)$ for some ϵ. With Therefore, we omit the tedious description of this situation which does not impair our conclusion.

$$
\begin{aligned}
& V\left(E_\theta, \theta\right)-t_\theta-V\left(E_{\theta^{\prime}}, \theta\right)+t_{\theta^{\prime}} \\
= & -\theta \pi_2(\theta)-2 \bar{\theta} t_\theta+2 \bar{\theta} t_{\theta^{\prime}}+\theta \pi_2\left(\theta^{\prime}\right) \\
= & \theta(\pi_2\left(\theta^{\prime}\right)-\pi_2(\theta))+2 \bar{\theta}(\frac{\theta^{\prime}}{2 \bar{\theta}}\left(1-\pi_2\left(\theta^{\prime}\right)\right)-\frac{\theta}{2 \bar{\theta}}(1-\pi_2(\theta))+\frac{\int_{\theta^{\prime}}^\theta(1-\pi_2(t)) d t}{2 \bar{\theta}}) \\
= & \left(\theta^{\prime}-\theta\right)\left(1-\pi_2\left(\theta^{\prime}\right)\right)+\int_{\theta^{\prime}}^\theta(1-\pi_2(t)) d t \\
= & -\left(\theta^{\prime}-\theta\right) \pi_2\left(\theta^{\prime}\right)+\int_\theta^{\theta^{\prime}} \pi_2(t) d t \\
\geqslant & 0
\end{aligned}
$$

ii.Non-Responsive IC

IC$[\theta \rightarrow \theta^{\prime}]$ where $E_{\theta^{\prime}}$ is not responsive for θ

$$
\begin{aligned}
V\left(E_\theta, \theta\right)-t_\theta-V\left(E_{\theta^{\prime}}, \theta\right)+t_{\theta^{\prime}} & =-\theta \pi_2(\theta)-2 \bar{\theta} t_\theta-2 m t^*+\frac{1}{2} \pi_1\left(\theta^{\prime}\right)+t_{\theta^{\prime}} \\
& =-\theta \pi_2(\theta)-2 \bar{\theta} t_\theta+(1-2 m) t^* \\
& =2 \bar{\theta}\left(t^*-t_\theta\right)-\theta \pi_2(\theta) \\
& =\bar{\theta} \pi_1(\theta)-\theta \pi_2(\theta) \\
& \geqslant 0 .
\end{aligned}
$$

Individual Rationality

For $\theta>\lambda(\underline{\theta})$

$$
\begin{gathered}
V(E_\theta, \theta)-t_\theta=\theta+m-m \pi_1(\theta)-\theta \pi_2(\theta)-t_\theta \geq V(E_{\underline{\theta}}, \theta)=\theta+m-\tfrac{1}{2} \pi_1(\underline{\theta})-t_{\underline{\theta}} \geq \\
\underline{\theta}+m-m \pi_1(\underline{\theta})-t_{\underline{\theta}}=V(E_{\underline{\theta}}, \underline{\theta}) \geq 0
\end{gathered}
$$

The first inequality is from the IC$[\theta \rightarrow \underline{\theta}]$. The second inequality is from $\theta>\lambda(\underline{\theta})$, and the last inequality is from IR$[\underline{\theta}]$.

For $\theta \leq \lambda(\underline{\theta})$

$$
\begin{gathered}
V(E_\theta, \theta)-t_\theta=\theta+m-m \pi_1(\theta)-\theta \pi_2(\theta)-t_\theta \geq V(E_{\underline{\theta}}, \theta)=\theta+m-m \pi_1(\underline{\theta})-\theta \pi_2(\underline{\theta})-t_{\underline{\theta}} \geq \\
\underline{\theta}+m-\underline{\theta} \pi_2(\underline{\theta})-m \pi_1(\underline{\theta})-t_{\underline{\theta}}=V(E_{\underline{\theta}}, \theta) \geq 0
\end{gathered}
$$

The first inequality is from the IC$[\theta \rightarrow \underline{\theta}]$. The second inequality is from the monotonicity, and the last inequality is from IR$\underline{\theta}$.

So all IR$[\theta]$ can be reduced to IR$[\underline{\theta}]$ and IC$[\theta \rightarrow \underline{\theta}]$

Responsiveness

For θ

$$\overline{\theta}\frac{\pi_1(\theta)}{\pi_2(\theta)} = 2\overline{\theta}\frac{t^* - t_\theta}{\pi_2(\theta)} \quad = 2\overline{\theta}\frac{\frac{\overline{\theta}-\theta+\theta\pi_2(\theta)}{2\overline{\theta}} - \frac{\int_\theta^{\overline{\theta}}(1-\pi_2(t))dt}{2\overline{\theta}}}{\pi_2(\theta)} = \frac{\theta\pi_2(\theta) + \int_\theta^{\overline{\theta}}\pi_2(t)dt}{\pi_2(\theta)}$$

$$\geqslant \frac{\theta\pi_2(\theta)}{\pi_2(\theta)}$$

$$= \theta$$

So far we have proved the sufficiency.

B.6 Proof of Lemma 10 and 11

$$\max_{\pi_2(\theta),\pi_1(\theta),t_\theta} \int_{\underline{\theta}}^{\overline{\theta}} t_\theta dF(\theta)$$

s.t.

$$\begin{aligned}
&\pi_2(\theta) : \Theta \to [0,1] \text{ is non-increasing} \\
&t_\theta = \tfrac{\theta}{2\overline{\theta}}(1-\pi_2(\theta)) + t_{\underline{\theta}} - \tfrac{\int_{\underline{\theta}}^{\theta}(1-\pi_2(t))dt}{2\overline{\theta}} \\
&t_\theta + \tfrac{1}{2}\pi_1(\theta) = t^* \\
&m - m\pi_1(\underline{\theta}) - t_{\underline{\theta}} \geq 0
\end{aligned}$$

And [23]

$$\begin{aligned}
&\int_{\underline{\theta}}^{\overline{\theta}} t_\theta dF(\theta) \\
=&\int_{\underline{\theta}}^{\overline{\theta}} \left(t_{\underline{\theta}} + \frac{\theta}{2\overline{\theta}}(1-\pi_2(\theta)) - \frac{\int_{\underline{\theta}}^{\theta}(1-\pi_2(t))\,dt}{2\overline{\theta}} \right) dF(\theta) \\
=&\frac{1}{2\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \left(\frac{1-F(\theta)}{f(\theta)} - \theta \right) \pi_2(\theta) dF(\theta) + \overline{\theta} t_{\underline{\theta}} \\
=&\frac{1}{2\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \left[\int_\theta^{\overline{\theta}} (1 - F(t) - tf(t))dt \right] d\pi_2(\theta) + \overline{\theta} t_{\underline{\theta}}
\end{aligned}$$

It is also easy to verify that $m = m\pi_1(\underline{\theta}) + t_{\underline{\theta}}$. Therefore, given the existence of θ^* and the formulation $t_\theta = \frac{\theta}{2\overline{\theta}}(1-\pi_2(\theta)) + t_{\underline{\theta}} - \frac{\int_{\underline{\theta}}^{\theta}(1-\pi_2(t))dt}{2\overline{\theta}}$, we have

$$\begin{aligned}
&t_\theta = \tfrac{1}{2\overline{\theta}}\left(1-\pi_2(\theta)\right) + t_{\underline{\theta}} - \tfrac{\int_{\underline{\theta}}^{\theta}(1-\pi_2(t))dt}{2\overline{\theta}} \\
&t^* = \tfrac{1}{2\overline{\theta}} + t_{\underline{\theta}} - \tfrac{\int_{\underline{\theta}}^{\overline{\theta}}(1-\pi_2(\theta))dt}{2\overline{\theta}} \\
&\pi_1(\underline{\theta}) = \tfrac{1}{\overline{\theta}} - \tfrac{\int_{\underline{\theta}}^{\overline{\theta}}(1-\pi_2(\theta))dt}{\overline{\theta}} \\
&t_{\underline{\theta}} = m - \tfrac{m}{\overline{\theta}} + \tfrac{m\int_{\underline{\theta}}^{\overline{\theta}}(1-\pi_2(\theta))dt}{\overline{\theta}}
\end{aligned}$$

[23] Here we use the fact that $\int_a^b g(\theta)x(\theta)d\theta = \int_{\underline{\theta}}^{\bar{\theta}} \mathbf{1}_{\{\theta\leq b\}} \left(\int_{\max\{a,\theta\}}^b g(\tau)d\tau\right) dx(\theta)$.

$$
\begin{aligned}
&\frac{1}{2\overline{\theta}}\int_{\underline{\theta}}^{\overline{\theta}}\left[\int_{\theta}^{\overline{\theta}}(1-F(t)-tf(t))dt\right]d\pi_2(\theta)+\bar{\theta}m\left[1-\frac{1}{\bar{\theta}}+\frac{\int_{\underline{\theta}}^{\overline{\theta}}(1-\pi_2(\theta))\,dt}{\bar{\theta}}\right]\\
=&\frac{1}{2\overline{\theta}}\int_{\underline{\theta}}^{\overline{\theta}}\left[\int_{\theta}^{\overline{\theta}}(1-F(t)-tf(t))dt\right]d\pi_2(\theta)+\bar{\theta}m(2-\frac{1}{\bar{\theta}})-m\int_{\underline{\theta}}^{\overline{\theta}}\pi_2(\theta)d\theta\\
=&\int_{\underline{\theta}}^{\overline{\theta}}\{\frac{1}{2\overline{\theta}}\left[\int_{\theta}^{\overline{\theta}}(1-F(t)-tf(t))dt\right]-m(\overline{\theta}-\theta)\}d\pi_2(\theta)+\bar{\theta}m(2-\frac{1}{\bar{\theta}})
\end{aligned}
$$

Therefore, we can rewrite the optimization problem as

$$\max_{\pi_2(\theta)}\int_{\underline{\theta}}^{\overline{\theta}}\Phi(\theta)d\pi_2(\theta)$$

s.t.

$$\pi_2(\theta):\Theta\rightarrow[0,1]\text{ is non-increasing}$$

where $\Phi(\theta)=\frac{1}{2\overline{\theta}}\left[\int_{\theta}^{\overline{\theta}}(1-F(t)-tf(t))dt\right]-m(\overline{\theta}-\theta)$.

The optimization problem consists of maximizing a linear functional subject to a non-increasing function ranging from $[0,1]$[24]. By the infinite-dimensional extension of Carathéodory's theorem[25], it follows that there exists an optimal allocation rule that is one of the extreme points of the set of non-increasing functions ranging from $[0,1]$, where $im\pi_2\subseteq\{0,1\}$[26]. With the previous conclusions, we know that $\pi_2^*(\theta)=1$ on $\theta\in[\underline{\theta},\theta^*)$ while $\pi_2^*(\theta)=0$ on $\theta\in[\theta^*,\overline{\theta}]$ for some θ^* .

Given the functional form of $\pi_2^*(\theta)$, $t_\theta=\frac{\theta}{2\overline{\theta}}(1-\pi_2(\theta))+t_{\underline{\theta}}-\frac{\int_{\underline{\theta}}^{\theta}(1-\pi_2(t))dt}{2\overline{\theta}}$, and the tiered-pricing structure of the optimal mechanism, we can derive that the optimal mechanism features two tiers with $\lambda(\theta)=\theta^*$ for any $\theta\in[\underline{\theta},\theta^*)$.

Therefore, we can further rewrite the optimization as choosing an optimal two-tier pricing mechanism. The designer selects an optimal threshold θ^* to maximize her revenue.

$$\max_{\theta^*,t_{\underline{\theta}},t^*}t_{\underline{\theta}}F(\theta^*)+t^*(1-F(\theta^*))$$

s.t.

$$
\begin{aligned}
&m-m\pi_1(\underline{\theta})-t_{\underline{\theta}}=0\\
&t_{\underline{\theta}}+\tfrac{1}{2}\pi_1(\underline{\theta})=t^*\\
&\pi_1(\underline{\theta})=\tfrac{\theta^*}{\overline{\theta}}
\end{aligned}
$$

[24]Formally, if π_2^i is the solution to optimization, for $i\in\{1,2\}$, then $\alpha\pi_2^1+(1-\alpha)\pi_2^2$ is a solution to problem, for any $\alpha\in(0,1)$.

[25]See more details in C.1.2

[26]See more details in C.1.1

Substitue these components and we have

$$\begin{aligned}
&mF(\theta^*) - m\pi_1(\underline{\theta}) F(\theta^*) + \left[m + \left(\tfrac{1}{2} - m\right)\pi_1(\underline{\theta})\right](1 - F(\theta^*)) \\
&= m + \left(\tfrac{1}{2} - m\right)\pi_1(\underline{\theta}) - \left(\tfrac{1}{2} - m\right)\pi_1(\underline{\theta}) F(\theta^*) - m\pi_1(\underline{\theta}) F(\theta^*) \\
&= m + \pi_1(\underline{\theta})\left(\overline{\theta} - \tfrac{1}{2}F(\theta^*)\right) \\
&= m + \tfrac{\theta^*}{\overline{\theta}}\left(\overline{\theta} - \tfrac{1}{2}F(\theta^*)\right)
\end{aligned}$$

Therefore the optimal mechanism is that

$$\begin{gathered}
\theta^* \in \arg\max_\theta \theta\left(\overline{\theta} - \tfrac{1}{2}F(\theta)\right) \\
\pi_1(\underline{\theta}) = \tfrac{\theta^*}{\overline{\theta}} \text{ and } \pi_2(\underline{\theta}) = 1
\end{gathered}$$

C Appendix: Completion of the Omitted Details in Proof

C.1 Completion of the proof of Theorem 2

In the proof of theorem 2, we transform the optimization into maximizing a linear functional subject to a non-increasing function. Here we refer to an infinite-dimensional extension of Carathéodory's theorem found in Kang (2023).

Here we state the omitted proof of two conclusions to complete the proof of theorem 2, where we refer to the proof in Kang (2023).

C.1.1 Extreme Points of $\Pi = \{\pi | \pi : \Theta \to [0,1],\ \pi \text{ is non-increasing}\}$

Denote $\mathcal{P} = \{\pi | \pi : \Theta \to [0,1],\ \pi \text{ is non-increasing, and } \operatorname{im}\pi \subseteq \{0,1\}\}$.

Proof of $\mathcal{P} \subseteq \mathrm{ex}\Pi$

Suppose that $\pi \in \mathcal{P}$, and $\pi = \alpha\pi_1 + (1-\alpha)\pi_2$ for $\pi_1, \pi_2 \in \Pi$ and $\alpha \in (0,1)$. Then $\alpha\pi_1(\theta) + (1-\alpha)\pi_2(\theta) = \pi(\theta) \in \{0,1\}$ for almost every $\theta \in [\underline{\theta}, \overline{\theta}]$, which implies that $\pi_1 = \pi_2 = \pi$ for almost everywhere on Θ. Hence $\pi \in \mathrm{ex}\,\Pi$.

Proof of $\mathrm{ex}\Pi \subseteq \mathcal{P}$

For any $\pi \in \Pi$ satisfying $m(\{\theta | x(\theta) \notin \{0,1\}\}) > 0$, where m is the Lebesgue measure. Define $\pi_1, \pi_2 : \Theta \to [0,1]$ by $\pi_1 = \pi^2$ and $\pi_2 = 2\pi - \pi^2$; by construction, π_1, $\pi_2 \in \Pi$ and $\pi = (\pi_1 + \pi_2)/2$. Note that $\pi_1 \neq \pi_2$. Therefore $\pi = (\pi_1 + \pi_2)/2$ where $\pi_1, \pi_2 \in \Pi$ are distinct; hence $\pi \notin \mathrm{ex}\Pi$.

Therefore $\mathcal{P} = \mathrm{ex}\Pi$.

C.1.2 An Infinite-dimensional Extension of Carathéodory Theorem

Theorem 3. *Let K be a convex, compact set in a locally convex Hausdorff space, and let $l : K \to \mathcal{R}^m$ be a continuous affine function such that $\sum \subseteq \operatorname{im} l$ is a closed and convex set. Suppose that*

$l^{-1}(\sum)$ is nonempty and and that $\Omega : K \to \mathcal{R}$ is a continuous convex function. Then there exists $z^* \in l^{-1}(\sum)$ such that $\Omega(z^*) = \max_{z \in l^{-1}(\sum)} \Omega(z)$ and

$$z^* = \textstyle\sum_{i=1}^{m+1} \alpha_i z_i, \textit{ where } \sum_{i=1}^{m+1} \alpha_i = 1, \textit{ and for all } i,\ \alpha_i \geq 0,\ z_i \in \mathrm{ex}K$$

Proof. See in Bauer (1958) and Szapiel (1975). □

Kang (2023) shows that these three conditions (convexity, compact in the L_1 topology, and the existence of the optimal mechanism) are satisfied in a mechanism design with transferable utility setting Therefore we can apply Carathéodory Theorem to our problem.

C.2 Completion of Lemma 7, 10 and 11

To guarantee the validity of the transitivity (of monotonicity) across different $[\theta, \lambda(\theta))$ in lemma 7, and the existence of the intervals in lemma 10 and 11, it remains to exclude one situation where for all $\hat{\theta} \in [\theta, \lambda(\theta))$, $\lambda(\hat{\theta}) = \lambda(\theta)$ when $\lambda(\theta) \neq \lambda(\lambda(\theta))$.

IC$[\hat{\theta} \to \lambda(\theta)]$ holds for all $\hat{\theta} \in [\theta, \lambda(\theta))$, which requires that

$$V(E_\theta, \lambda(\theta)) - t_\theta \geq V(E_{\lambda(\theta)}, \lambda(\theta)) - t_{\lambda(\theta)}$$

Combined with IC$[\lambda(\theta) \to \theta]$, we have

$$-\tfrac{1}{2}\pi_1(\theta) - t_\theta = -\lambda(\theta)\pi_2(\lambda(\theta)) - m\pi_1(\lambda(\theta)) - t_{\lambda(\theta)}$$

Combining the two facts (i) IC$[\gamma(\theta) \to \theta]$ is binding while IC$[\gamma(\theta) \to \lambda(\theta)]$ may not , and (ii) IC$[\gamma(\lambda(\theta)) \to \lambda(\theta)]$ is binding while IC$[\gamma(\lambda(\theta)) \to \theta]$ may not, we have

$$-\tfrac{1}{2}\pi_1(\theta) - t_\theta = -\tfrac{1}{2}\pi_1(\lambda(\theta)) - t_{\lambda(\theta)}$$

By $\lambda(\lambda(\theta)) > \lambda(\theta)$, we have

$$-\tfrac{1}{2}\pi_1(\theta) - t_\theta = -\lambda(\theta)\pi_2(\lambda(\theta)) - m\pi_1(\lambda(\theta)) - t_{\lambda(\theta)} > -\tfrac{1}{2}\pi_1(\lambda(\theta)) - t_{\lambda(\theta)} = -\tfrac{1}{2}\pi_1(\theta) - t_\theta$$

So it is contradictory.

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